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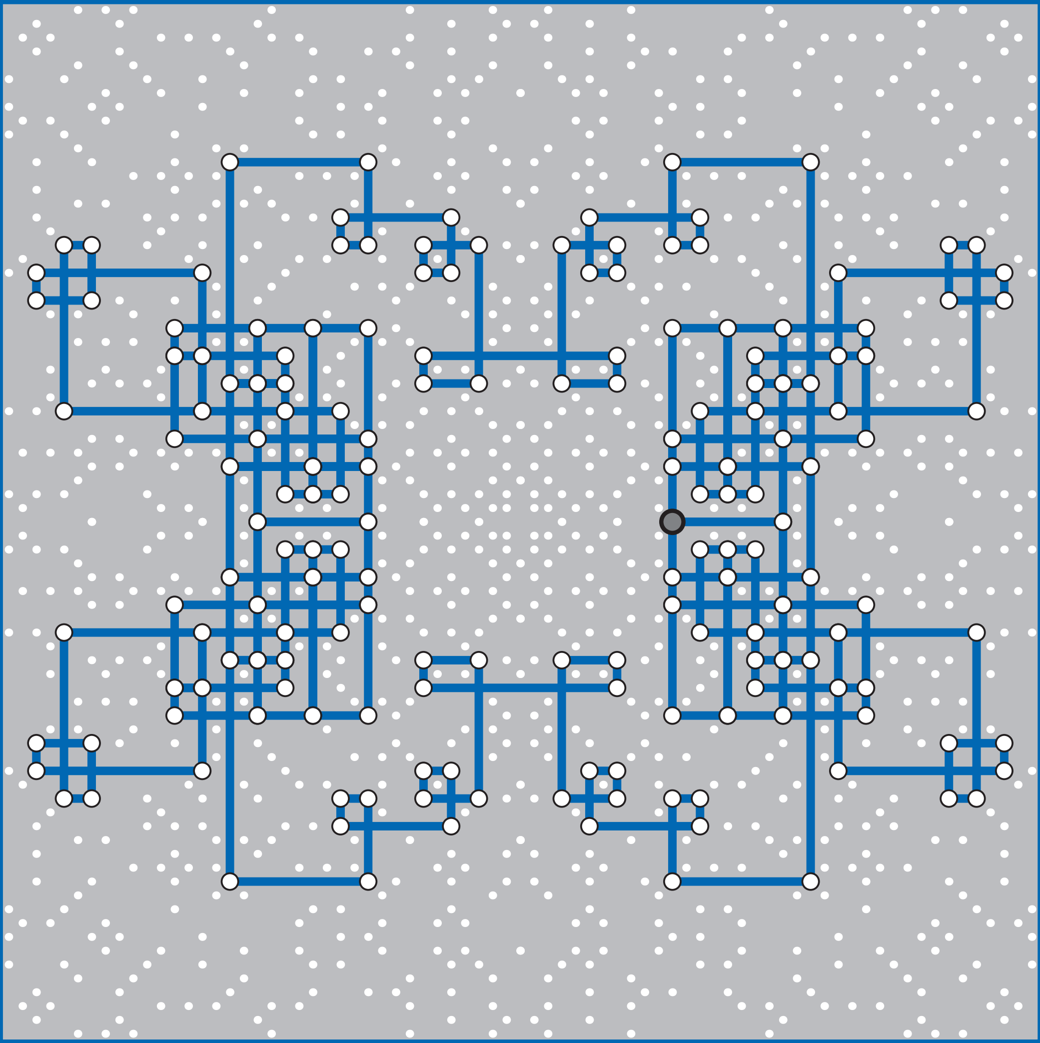
MATHEMATICS MAGAZINE



Gaussian Prime Spiral (page 14)

- Parametrizing an ellipse with constant speed
- Infinite products
- Sitting down for dinner at a twin convention
- The Lah numbers

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Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the *Magazine*. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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Artist's Credit: Gaussian prime spiral, by Joseph O'Rourke and Stan Wagon. For a description, see page 14.

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LETTER FROM THE EDITOR

I recently attended a dinner celebration whose hosts partitioned the n guests into k groups, one group for each table. In how many ways could they have done that, if the only requirement was that no table be empty? The answers, for each n and k, are given by the *Stirling Numbers of the Second Kind.* We have mentioned these numbers in this space before. I warned you that you would be seeing them everywhere!

As far as I know, we were all distinguishable people. But what if we had all been pairs of indistinguishable twins? Then the hosts might have sought help from Martin Griffiths, whose article in this issue is called “Sitting Down for Dinner at a Twin Convention.” In short, he shows us how to count the partitions of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ into k non-empty parts.

Stirling Numbers are also used with rising powers, $x^{\overline{k}} = x(x+1)\cdots(x+k-1)$, and falling powers, $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$. In combinatorics, these are often more useful than ordinary powers. But you’ll need to translate among them. You can write a rising power in terms of ordinary powers using Stirling Numbers of the First Kind, and you can write an ordinary power as a combination of falling powers using Stirling Numbers of the Second Kind. Or, you can skip a step and write a rising power in terms of falling powers, using the *Lah Numbers,* which are the subject of an article in this issue by Daboul, Mangaldan, Spivey, and Taylor. The authors did their work through a web forum, and have never met each other!

Flight plans meet elliptic integrals in the article by Michelle Ghrist and Eric Lane, who have a professional interest in both. The problem arose when an Air Force Academy cadet was asked to plot an elliptical path for an aircraft flying at constant speed. It is arc-length parametrization in a practical environment.

Are you intrigued by Viète’s formula,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots?$$

The article by Samuel Moreno and Esther García shows how to prove it, and how you can find more infinite products like it.

In the Notes Section, Alan Beardon grapples with a delay differential equation. The power series method gives solutions, but are they the only solutions? Richard Beals also discusses the power series method. Its scope is limited, but it reaches most of the differential equations that define “special functions.”

Finally, will you be celebrating Pi Day next month? Are you distressed that the date 3/14 is just an approximation to Pi Day? Then Mark Lynch has found one way to set your mind at ease.

Thanks to Stan Wagon and Joseph O’Rourke for providing this month’s cover image; look for its description (and a conjecture) inside the issue. Thanks, too, to the Putnam committee for giving us a first look at the problems and solutions from December’s Putnam Examination.

Walter Stromquist, Editor

ARTICLES

Be Careful What You Assign: Constant Speed Parametrizations

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This exploration began, as many good problems do, with a student's question. This is a story about how interesting problems can arise quite accidentally and how a perceptive and persistent student can help us explore something new.

Several years ago, Eric Lane came to me (Michelle Ghrist) with a challenge. He was trying to find a new way to parametrize an ellipse; instead of the usual parametrization $x(t) = a\cos(\omega t)$, $y(t) = b\sin(\omega t)$, he was trying to find a parametrization with a constant speed. Eric was simultaneously taking my differential equations course and a multivariable calculus course, and he was working on a calculus project in which students were given a terrain map and asked to plot a pathway for an aircraft over a target in a valley. Given that the plane needs to fly at a constant 200 knots and cannot make sharp turns (i.e., no cusps allowed), the students were asked to parametrize their path using only linear, circular, and/or elliptical segments and then analyze the acceleration of the aircraft for their chosen path.

Eric had realized that parametrizing an ellipse in the usual way would not give a constant tangential speed as specified by the project instructions. Upon talking to the calculus instructors, I discovered that they had overlooked this fact; they meant for the students to use either the standard ellipse parametrization or only linear and circular segments. However, Eric was persistent, wanting to follow the instructions explicitly; his enthusiasm might have also been fueled by the extra credit his calculus instructor offered if he could find a better way to parametrize elliptical segments.

After trying out some ideas with Eric, I realized that this was no easy task. So, we had found a challenge: Parametrize an ellipse with constant speed and then explore how this parametrization compares with the standard one. Interestingly enough, our solution method ended up relying heavily on differential equations.

A simpler problem: parametrizing a circle

We begin with the familiar case of parametrizing a circle of radius $R > 0$ centered at the origin using constant speed C. We examine this case in order to set up a similar treatment of ellipses later. The Cartesian equation of our circle is

$$x^2 + y^2 = R^2. \tag{1}$$

Math. Mag. **86** (2013) 3–14. doi:10.4169/math.mag.86.1.003.

We let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ specify the position of the aircraft at any time t. Because the speed is a constant C, we have $|\mathbf{r}'(t)| = C$ and thus

$$(x'(t))^2 + (y'(t))^2 = C^2. \tag{2}$$

Differentiating (1) with respect to t and solving for $x'(t)$ gives

$$x'(t) = -\frac{y}{x}y'(t). \tag{3}$$

We then substitute (3) into (2) and simplify to find

$$(y'(t))^2\left(1 + \frac{y^2}{x^2}\right) = C^2.$$

Rewriting this using (1) and taking the square root of both sides gives

$$\frac{1}{\sqrt{R^2 - y^2}}\frac{dy}{dt} = \pm\frac{C}{R}. \tag{4}$$

Integrating both sides of (4) with respect to t gives

$$\arcsin\frac{y}{R} = \pm\frac{C}{R}t + \alpha,$$

which gives the solution

$$y(t) = R\sin\left(\pm\frac{C}{R}t + \alpha\right). \tag{5}$$

Substituting this result into (1), simplifying, and solving for $x(t)$ gives

$$x(t) = \pm R\cos\left(\pm\frac{C}{R}t + \alpha\right). \tag{6}$$

Here, α is a constant that is determined by the initial condition and the direction of travel around the circle (clockwise vs. counter-clockwise); the direction of travel is also affected by the sign choices. For example, suppose that $t_0 = 0$, the initial position is $(R, 0)$, and we restrict $0 \leq \alpha < 2\pi$. Then $\mathbf{r}(0) = \langle x(0), y(0)\rangle = \langle \pm R\cos\alpha, R\sin\alpha\rangle = \langle R, 0\rangle$, which has two possible solutions: $\alpha = 0$ and $\alpha = \pi$. For $\alpha = 0$, we must choose $+$ for the initial sign in (6); in this case, the choice of $+$ for the other sign choice in (5) and (6) leads to counter-clockwise motion. For $\alpha = \pi$, we must choose $-$ for the initial sign in (6), in which case the choice of $+$ for the other sign leads to clockwise motion. By including horizontal and vertical shifts, this solution can easily be adapted for circles not centered at the origin.

After a brief interlude, we will generalize this approach to ellipses. We challenge the reader to attempt to parametrize the ellipse $x^2/a^2 + y^2/b^2 = 1$ with constant speed before reading on. We will find that the relatively simple approach of this section, when applied to ellipses, will lead us down a path requiring a special function known as the elliptic integral of the second kind; we will also need the inverse of this special function. Thus, before proceeding, we provide some background information on elliptic integrals.

Background: elliptic integrals

Elliptic integrals belong to a category of functions that are defined using integrals because they cannot be expressed directly in terms of more elementary functions. (Other functions defined in this way include the natural logarithmic function and the error function.) Elliptic integrals appear quite frequently in applications involving ellipses such as mechanics, electrodynamics, and astronomy.

While there are several different kinds (and more general definitions) of elliptic integrals (see, for example, [**10**]), here we focus on the *elliptic integral of the second kind;* it is defined by

$$E(\phi \mid m) \equiv \int_0^{\phi} \sqrt{1 - m \sin^2 \mu}\, d\mu \tag{7}$$

or equivalently by

$$E(\phi \mid m) \equiv \int_0^{\sin\phi} \sqrt{\frac{1 - mz^2}{1 - z^2}}\, dz, \tag{8}$$

where the two definitions are related via the substitution $z = \sin\mu$. The constant $m \in [-1, 1]$ is a parameter, while the independent variable ϕ is often referred to as the *amplitude.* Several different notations are used for elliptic integrals. For example, the common notation $E(\phi, k)$ reflects the substitution $m = k^2$ in (7) and (8), where $k \in [0, 1]$ is the *elliptic modulus.* We avoid this substitution, as we will need to consider negative values of the parameter m.

For fixed parameter m, the elliptic integral of the second kind is an increasing function of ϕ. FIGURE 1 shows the graphs of $E(\phi \mid 0.9)$ and $E(\phi \mid -0.9)$. The graph of $E(\phi \mid 0) = \phi$ is a straight line. As m approaches ± 1, the graph deviates more from this line; this behavior is particularly evident for positive values of m.

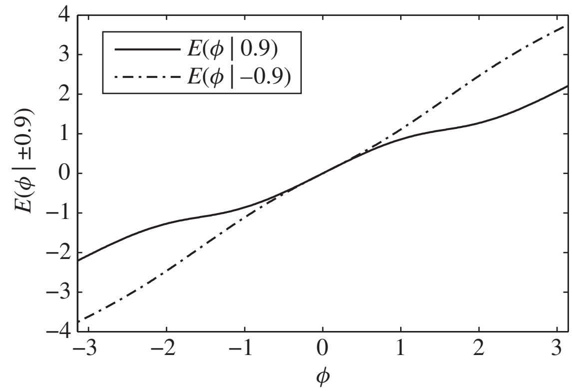

Figure 1 Graph of the elliptic integral of the second kind $E(\phi \mid m)$ for $m = \pm 0.9$.

We will soon need the inverse of the elliptic integral of the second kind (meaning the inverse with respect to the amplitude, ϕ, keeping the parameter m fixed). From FIGURE 1, it appears that $E(\phi \mid m)$ is invertible with respect to amplitude; we can also see this from the definition, as the integrands in (7) and (8) are strictly positive except possibly at isolated points. Much information appears in the literature about inverses of elliptic integrals of the first kind (sometimes termed *Jacobi elliptic functions*). However, very little appears about inverses of elliptic integrals of the second kind; we know of no special name given to this function.

We will also need the *complete elliptic integral of the second kind,* which specifies a value of $\pi/2$ for the amplitude ϕ in the definition of the elliptic integral:

$$E(m) \equiv E\left(\frac{\pi}{2} \mid m\right) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\mu}\, d\mu = \int_0^1 \sqrt{\frac{1 - mz^2}{1 - z^2}}\, dz. \tag{9}$$

This function frequently appears when expressing various geometric properties of an ellipse, such as the perimeter.

For more mathematical details on elliptic integrals and elliptic functions, see, for example, the historic texts [**1, 3, 7, 8**] or the more modern texts [**9, 13, 16**]. For details on applications of elliptic integrals and functions, see, for example, [**8, 9, 10**] or advanced mathematical physics texts. Finally, to explore the rich historical development of elliptic integrals and functions by such greats as Legendre, Gauss, Abel, Jacobi, Wallis, Newton, and Weierstrass, see [**4, 11, 14**], for example.

More difficult: parametrizing an ellipse

Armed with this knowledge of elliptic integrals of the second kind, we are now ready to consider the problem of finding a constant-speed parametrization of an ellipse. Without loss of generality, we consider an ellipse centered at the origin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{10}$$

where $a \geq b > 0$, so that the x-axis is the major axis.

Differentiating (10) with respect to t and solving for $x'(t)$ gives

$$x'(t) = -\frac{a^2}{b^2}\frac{y}{x}y'(t).$$

We then substitute this result into (2), the constant speed requirement, and simplify to find

$$(y'(t))^2\left(1 + \frac{a^4}{b^4}\frac{y^2}{x^2}\right) = C^2, \tag{11}$$

where $C > 0$ is the speed of the object. Rewriting this equation using (10) and taking the square root of both sides gives

$$\sqrt{\frac{b^2 - \left(\frac{e^2}{e^2-1}\right)y^2}{b^2 - y^2}}\,\frac{dy}{dt} = \pm C, \tag{12}$$

where $e = \sqrt{1 - b^2/a^2}$ is the eccentricity of the ellipse. This is a separable first-order differential equation for $y(t)$. Note that if $e = 0$ (that is, if we have a circle where $a = b = R$), equation (12) is equivalent to (4).

We make the substitution $z = y/b$ in (12) to find

$$\sqrt{\frac{1 - \left(\frac{e^2}{e^2-1}\right)z^2}{1 - z^2}}\,\frac{dz}{dt} = \pm\frac{C}{b}. \tag{13}$$

We now observe that the left-hand side of (13) is identical to the integrand in (8), where $m = \frac{e^2}{e^2-1} \leq 0$ because $0 \leq e < 1$. Therefore, integrating both sides of (13) with respect to t gives

$$E\left(\arcsin z(t) \mid \frac{e^2}{e^2-1}\right) = \pm\frac{C}{b}t + \beta.$$

Recalling that $y(t) = bz(t)$, we find the solution to (12) to be

$$y(t) = b\sin\left(u(t)\right), \tag{14}$$

where $u(t) = \arcsin(z(t))$ is given implicitly by

$$E\left(u(t) \mid \frac{e^2}{e^2-1}\right) = \pm\frac{C}{b}t + \beta. \tag{15}$$

Because $E\,(\phi \mid m)$ is an increasing function of ϕ for a given value of m, (15) defines a unique solution for $u(t)$. Effectively, $u(t)$ is an inverse for the elliptic integral.

Substituting these results into (10), simplifying, and solving for $x(t)$ gives

$$x(t) = \pm a\cos\left(u(t)\right), \tag{16}$$

where $u(t)$ is defined implicitly by (15).

As before, the constant β and sign choices in (15) and (16) are determined by the initial conditions and the direction of travel. If, for example, $(x(0), y(0)) = (a, 0)$, one solution is $\beta = 0$ with a sign choice of $+$ in (16). Then the sign choice of $+$ in (15) gives counterclockwise motion, for example.

For comparison, consider the standard counterclockwise parametrization of the ellipse (10) with initial condition $(x(0), y(0)) = (a, 0)$, which is given by

$$x(t) = a\cos(\omega t), \quad y(t) = b\sin(\omega t). \tag{17}$$

In general, the time needed for one traversal of an ellipse is given by the perimeter P divided by the average speed C. Thus $T = 2\pi/\omega = P/C$ and $\omega = 2\pi C/P$. For $a \geq b$, $P = 4aE(e^2) = 4bE(\frac{e^2}{e^2-1})$, where $E(m)$ is the complete elliptic integral defined by (9); this can be found by computing the arc length of the curve (to be further discussed in a later section). Therefore

$$\omega = \frac{\pi C}{2aE\left(e^2\right)} = \frac{\pi C}{2bE\left(\frac{e^2}{e^2-1}\right)}. \tag{18}$$

Example 1: Moderate eccentricity Consider the counterclockwise parametrization given by (14)–(16) for an ellipse with $a = 5$, $b = 3$, and corresponding eccentricity $e = 0.8$. Choosing a speed of $C = 2$, initial condition $(x(0), y(0)) = (5, 0)$, and both $+$ signs for counterclockwise motion, we find that $\beta = 0$.

Mathematica was used to generate points on the ellipse, starting with $t = 0$ and using $\Delta t = 0.2$. FIGURE 2(a) shows a plot of points on the ellipse. The 64 points are all approximately (and slightly under) 0.4 units apart, which is to be expected because the speed is $C = 2$, and we chose a spacing of $\Delta t = 0.2$. The time needed for one traversal of the ellipse is given by the perimeter divided by the speed, which is $T = 10E(16/25) = 6E(-16/9) \approx 12.7635$ for this example; in the graph, the last dot corresponds to $t = 12.6$, which explains why the two dots near $(5, 0)$ are slightly closer than all of the others.

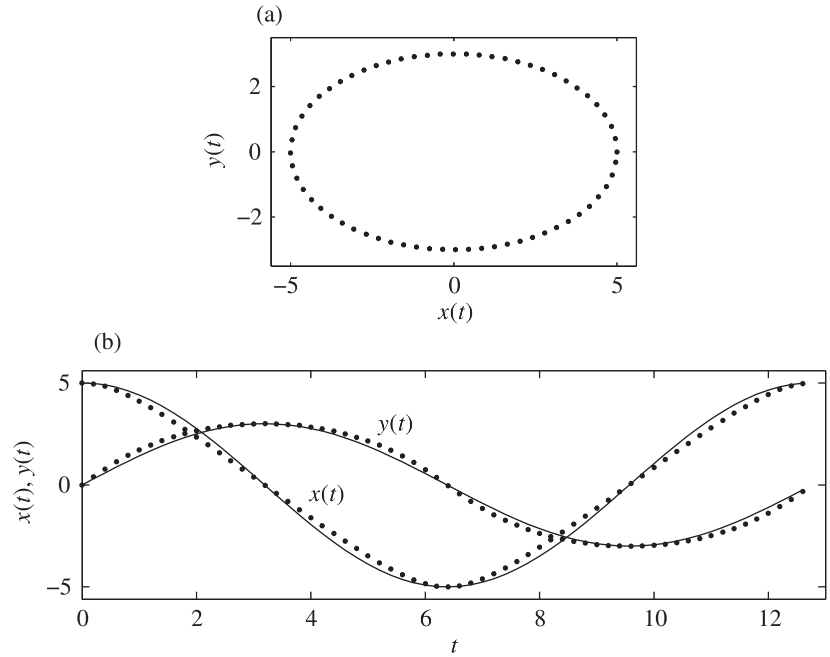

Figure 2 Comparison of new parametrization given by (14)–(16) (dotted) and standard parametrization given by (17) (solid), using $a = 5$, $b = 3$, $e = 0.8$, $C = 2$, and $\Delta t = 0.2$. For this case, the standard parametrization given by (17) has $\omega \approx 0.492$ from (18).

FIGURE 2(b) shows $x(t)$ and $y(t)$ versus t for one period of the ellipse as compared to the standard parametrization. From this, we can see the distortions of the new parametrization from the usual cosine and sine curves (shown as the solid curves); the curve associated with the minor axis (in this case, y) has less curvature near the extrema than a cosine curve, and the curve associated with the major axis has more curvature near the extrema than a sine curve.

The speed of the standard parametrization (17) is given by

$$|\omega| \sqrt{a^2 \sin^2 \omega t + b^2 \cos^2 \omega t} = a\, |\omega| \sqrt{1 - e^2 + e^2 \sin^2 \omega t}, \tag{19}$$

which ranges from $b\omega$ to $a\omega$. FIGURE 3 shows the speed as a function of time for the standard parametrization of an ellipse with average speed $C = 2$, as compared with the new parametrization (which is, as expected, a straight line). For this ellipse (with $e = 0.8$), the deviation of the standard parametrization from a constant speed is quite significant; the maximum deviation is approximately 26.2%.

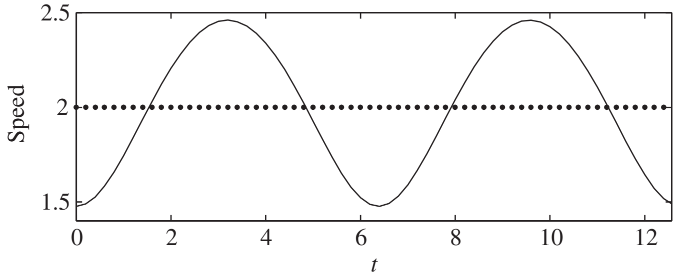

Figure 3 Speed for the two parametrizations with $e = 0.8$ (Example 1). The new parametrization has constant speed $C = 2$, while the speed for the standard parametrization is given by (19).

Example 2: Eccentricity close to one What happens to these two parametrizations if we increase the eccentricity of the ellipse? For the standard parametrization, as the eccentricity approaches one, the grid resolution near the ends of the minor axis becomes less dense, while the gridpoints near the ends of the major axis become more concentrated. This results in the speed of the standard parametrization being greater than the average speed near the ends of the minor axis, and significantly less than the average speed near the ends of the major axis.

This phenomenon is illustrated in FIGURES 4 and 5, which compare the two parametrizations for an ellipse with $e \approx 0.995$ and perimeter $P = 40E\,(0.99) \approx 40.64$. Note that in FIGURE 4(a), the spacing between the final and initial dots is

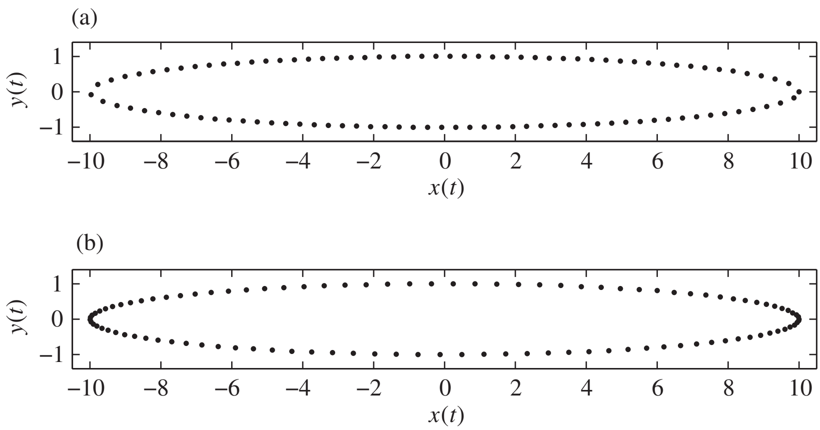

Figure 4 Ellipses with $e \approx 0.995$, using $\Delta t = 0.2$, $a = 10$, $b = 1$, and $C = 2$. (a) Constant speed parametrization defined by (14)–(16). (b) Standard parametrization given by (17), where $\omega \approx 0.309$ from (18). In both cases, 102 points were used.

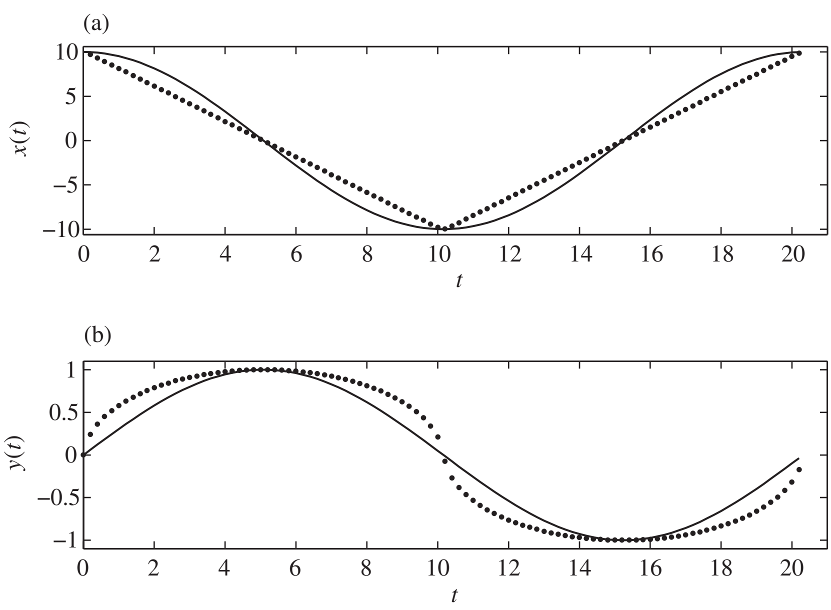

Figure 5 Comparing position functions for the two parametrizations for $e \approx 0.995$, standard (solid) and new (dotted). (a) x vs. t. The new parametrization is approaching piecewise linear. (b) y vs. t. The new parametrization is approaching two half-ellipses.

considerably less than all of the other spacings, because $T \approx 20.32$ and our final dot corresponds to $t = 20$.

FIGURE 5 shows that as the eccentricity of the ellipse approaches 1, the new parametrization of the major axis becomes closer to piecewise linear in t, while the new parametrization of the minor axis becomes more elliptical in t. This is logical because as the ellipse becomes more elongated, most of the movement occurs approximately parallel to the major axis, leading to the approximately piecewise-linear behavior in t. Then, due to the approximately piecewise behavior of $x(t)$ and the original equation of the ellipse (10), the equation for the minor-axis parametrization is necessarily approximately elliptical in t.

FIGURE 6 compares the speeds for the two parametrizations for this example. Note that, as e approaches 1, the speed of the standard parametrization given by (19) approaches $a\,|\omega \sin \omega t|$. The maximum deviation of standard parametrization's speed (19) from the average speed C is approximately 84.5% for this example.

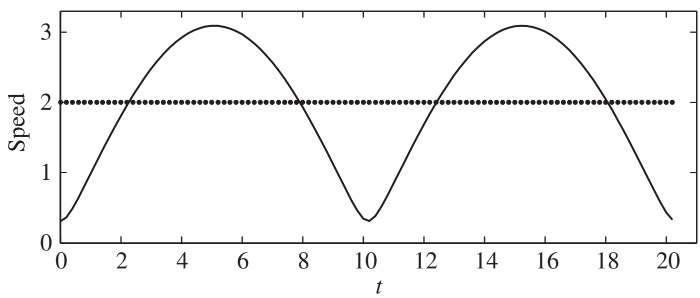

Figure 6 Speeds of the two parametrizations for $e \approx 0.995$. The new parametrization has constant speed; the speed of the standard parametrization is approaching $a\,|\omega \sin \omega t|$, which can be seen from (19).

Comparisons between the two parametrizations The maximum percentage deviation from average speed is a reasonable standard of comparison as it is unaffected by the average speed used and depends solely on the eccentricity of the ellipse. Using (19) and (18), we find that the maximum percentage deviation of the standard parametrization from constant speed as a function of eccentricity is given by

$$100\left(\frac{C - \omega \min(a, b)}{C}\right) = 100\left(1 - \frac{\pi\sqrt{1 - e^2}}{2E(e^2)}\right).$$

FIGURE 7 shows this maximum percentage deviation from average speed as a function of eccentricity. As the eccentricity approaches 1, the maximum deviation approaches 100%.

Now that we have a better understanding of our two parametrizations, we can discuss the advantages and disadvantages of each in light of the original calculus assignment. For ellipses with eccentricity less than about 0.8, chances are that an airplane pilot would not experience much difference between the two parametrizations, other than a constant speed versus one that varies by up to about 26%.

However, for ellipses with eccentricity closer to one, an airplane traveling using the standard parametrization would spend most of its time near the ends of the major axis. It would experience a large range of speeds, moving slower near the ends of the major axis and significantly faster near the ends of the minor axis; potential problems for the airplane include stalls from insufficient speed or speeds exceeding the capabilities of the airplane or the maximum safe speed for structural integrity.

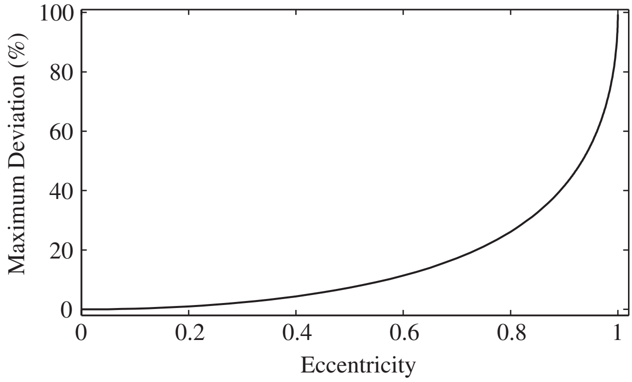

Figure 7 Maximum percentage deviation of the standard parametrization's speed from average speed as a function of eccentricity.

On the other hand, an airplane traveling using the new parametrization would travel at a constant speed and thus spend an equal amount of time everywhere. However, depending on the size of the ellipse, the plane might experience a sharp turn-around near the ends of the major axis, requiring a large amount of centripetal acceleration. This acceleration could possibly cause the aircraft to break apart or cause the pilot to experience large g-forces; another concern for the new parametrization is whether the engine could provide the thrust required to accomplish the mission.

Thus, a real pilot being required to travel on a highly eccentric elliptical path might, from common sense, take a moderate path between the two extremes of these parametrizations, avoiding the standard parametrization enough to prevent large variations in speed, but also avoiding the constant speed parametrization enough to prevent large centripetal accelerations.

Generalizations

There remain several questions of interest. What is the connection between these parametrizations and arc length? Can our methods be used to find constant speed parametrizations of all curves? Finally, do these methods find *all* possible parametrizations with constant tangential speed for the curves, or might there be others? In this section, we address these questions.

Arc length parametrizations The arc length of a curve is given by

$$s(t) = \int_{t_0}^{t} \left|\mathbf{r}'(\tau)\right| d\tau, \tag{20}$$

where $\mathbf{r}(t)$ specifies the position of an object at any time t. If an object is moving with constant speed C, then $s(t) = C\,(t - t_0)$, so $t = \frac{s}{C} + t_0$, allowing all of our previous parametrizations to be rewritten in terms of arc length.

In general, the arc length traveled along an ellipse as a function of the subtended angle θ is given by $s(\theta) = aE(\theta \mid e^2) = bE\left(\theta \mid \frac{e^2}{e^2-1}\right)$. This can be found from using (17) in (20). It thus makes sense that the constant-speed parametrization of an ellipse involves inverting an elliptic integral of the second kind.

One advantage of arc length parametrizations is that they are not dependent on speed. Parametrizations in terms of arc length are sometimes considered "ideal" parametrizations and have been studied extensively. For example, Gil [**6**] explored explicit arc length parametrizations of curves including ellipses and gave some applications, including computer graphics, curve rendering, and finding intersections of curves. Sakkalis and Farouki [**12**] showed that real plane curves other than straight lines cannot be parametrized by rational functions of arc length. In [**5**], Farouki presented a broader look at constant-speed parametrizations, including providing an alternate way to use elliptic integrals to parametrize ellipses that are not necessarily centered at the origin.

Other curves In what situations can our methods be used? We first observe that these methods can be used to parametrize curves other than circles and ellipses. For example, constant-speed parametrizations of hyperbolas can be found by replacing the sine and cosine functions in our derivations with hyperbolic sine and cosine functions. In addition, we can use our methods to find that the constant-speed parametrization of the parabola $y = Ax^2 + B$ is given implicitly by

$$\begin{aligned} \frac{x(t)}{2}\sqrt{1 + 4A^2\left(x(t)^2\right)} + \frac{1}{4A}\sinh^{-1}(2Ax(t)) &= \pm Ct + \delta, \\ y(t) &= A\left(x(t)\right)^2 + B, \end{aligned}$$

where C is the speed. The constant of integration δ and the sign choice are determined by the initial condition and the direction of motion.

For a more general case, we refer to Wilkinson [**15**], who considered constant-speed parametrizations of both plane and space curves. He showed that constant-speed parametrizations of plane curves of the form $F(x, y) = 0$ can be found by solving the following first-order system of differential equations for $x(t)$ and $y(t)$:

$$\begin{aligned} x'(t) &= \frac{\pm C F_y(x(t), y(t))}{\sqrt{(F_x(x(t), y(t)))^2 + \left(F_y(x(t), y(t))\right)^2}}, \\ y'(t) &= \frac{\mp C F_x(x(t), y(t))}{\sqrt{(F_x(x(t), y(t)))^2 + \left(F_y(x(t), y(t))\right)^2}}. \end{aligned} \tag{21}$$

One way to derive this system is by differentiating the Cartesian equation of the curve $F(x, y) = 0$ implicitly with respect to t, substituting the equation of constant speed (2), and then simplifying, much as we did for circles and ellipses. A different derivation of (21) is given in [**15**]. Wilkinson showed that in most cases, the solution to this system is unique other than direction of travel, which is determined by the sign choice; however, he did not demonstrate how to go about finding solutions as we have done here. As Wilkinson showed, there may not be a unique solution to this system in certain situations (e.g., singular points), and in many cases, this system will not have a closed form solution, requiring a numerical solver or an alternate solution method.

For example, for the ellipse (10), we have $F(x, y) = x^2/a^2 + y^2/b^2 - 1$, and (21) becomes

$$x'(t) = \frac{\pm Ca^2 y}{\sqrt{b^4x^2 + a^4y^2}}, \quad y'(t) = \frac{\mp Cb^2 x}{\sqrt{b^4x^2 + a^4y^2}}, \tag{22}$$

where the equation for $y'(t)$ is equivalent to (11). From here, one can decouple the system by using the equation of the curve (10) as we did previously for ellipses and then proceed to find a solution.

To establish uniqueness, consider theorems establishing the uniqueness of solutions of "nice" autonomous differential equations; see, for example [**2**, Theorem 7.1.1]. The only potential problem spot for the system (22) is at (0, 0), which is not on our ellipse. While our constant-speed parametrization of ellipses (14)–(16) that involves inverting elliptic integrals may be construed as inelegant, there is no other way to parametrize an ellipse with constant speed. However, there are other equally complicated ways to express the solution. For example, one alternative way to express this parametrization involves using Jacobi elliptic functions [**9**, Section 4.1]; however, this solution still requires inverting the elliptic integral of the second kind.

Our elliptical example, combined with Wilkinson's result, offers a way to determine which curves can be parametrized using our methods. It seems that these methods are applicable any time we can explicitly solve the Cartesian equation of the curve for x or y, although solving the resulting separable first-order differential equation may lead to complicated integrals, special functions, and/or implicit function definitions. We speculate that for curves that are only defined implicitly, we could apply Wilkinson's method (21) but would most likely not be able to reduce the resulting system to a single separable first-order differential equation.

Final thoughts

In our quest to create novel projects that are interesting to our students, we sometimes make things more complicated than we intend. Every so often, interesting mathematical questions arise somewhat accidentally. The question is: Do we encourage students (and ourselves) to pursue the unknown when the opportunity arises? Do we allow students to see glimpses of "real" mathematics, where problems don't necessarily have elegant solutions or can quickly increase in complexity as we deviate from the "canned" problems found in textbooks? Sometimes the thrill comes not from the final destination but from the journey.

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Summary We explore one aspect of a multivariable calculus project: parametrizing an elliptical path for an airplane that travels at constant speed. We find that parametrizing a constant speed elliptical path is significantly more complicated than a circular path and involves taking the inverse of the elliptic integral of the second kind. We compare our constant-speed parametrization to the standard ellipse parametrization ($x(t) = a\cos\omega t$, $y(t) = b\sin\omega t$) and generalize to parametrizing other constant-speed curves.

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About the Cover

Gaussian integers are complex numbers $a + bi$ with $a, b \in \mathbb{Z}$. A Gaussian integer is a *Gaussian prime* if it cannot be written as a product of other Gaussian integers, unless one factor is 1, -1, i, or $-i$. Thus 11, -11, $11i$, and $-11i$ are all Gaussian primes—but not 13, because 13 is the product of $3 + 2i$ and $3 - 2i$, both of which are Gaussian primes.

Gaussian prime spirals The white dots on the cover are Gaussian primes. Start at any Gaussian prime (the gray disk at 11 in the image shown). Walk east until you find another Gaussian prime (19 in the image). Turn left and walk north to the next Gaussian prime ($19 + 6i$ in the image). Again turn left and walk west until reaching a prime. Continue, always turning left at primes, until you return to your starting point via a southward step, thus entering a cycle. It is an open question whether such a cycle always forms. Even if, as expected, every horizontal line eventually reaches a Gaussian prime, it is not clear that every spiral must enter a loop. The loops can become very long: starting at $107 + 992i$ leads to a walk with 3,900,404 steps.

For a demonstration that allows one to dynamically move the starting point, see http://demonstrations.wolfram.com/GaussianPrimeSpirals. For further discussion see http://mathoverflow.net/questions/91423.

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New Infinite Products of Cosines and Viète-Like Formulae

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Viète's formula was the very first use of an infinite product in mathematics, and each of its factors contains nested square roots of 2, connected by plus signs. The formula is

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots, \tag{1}$$

and it was first derived in 1593 based on geometric considerations [**1, 10**]. Nowadays, the standard proof of Viète's formula relies on the iteration of the double angle formula $\sin 2x = 2\cos x \sin x$. For example, Simmons uses this method in *Calculus Gems* [**8**, pp. 252–253].

In this note we evaluate infinite products similar to the product in (1), with the novelty that some of the plus signs are replaced by minus signs. We call these products *Viète-like formulae*. To derive them, first we manipulate a simple trigonometric relation (other than the double angle formula) in order to obtain a family of infinite products of cosines. Next, with the aid of a formula of Servi [**7**, formula (7) on p. 328] (see also [**4**] for a generalization), we transform these infinite products of cosines into infinite products of nested square roots of 2. These Viète-like expressions turn out to represent numbers like π, $\sqrt{3}$, and $\sqrt{5-\sqrt{5}}$.

With our method we get beautiful formulae such as

$$\frac{2}{\sqrt{3}} = \sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{2-\sqrt{2-\sqrt{2}}}\cdots.$$

This particular formula is due to A. Levin [**2**, formula (50)], who gave a different proof. Our methods give many more formulae like this one.

Main idea and first result

To begin, let us recall the well-known formula

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b.$$

Choosing a and b such that $a+b=\pi/2$ and writing x for $a-b$, this identity transforms to

$$\cos x = 2\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right), \tag{2}$$

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which holds for every real x and can be considered as the key tool for developing our main results. We intend to iterate formula (2). We start by applying it with $(\pi/4 - x/2)$ in place of x to give a formula for $\cos(\pi/4 - x/2)$,

$$\cos\left(\frac{\pi}{4} - \frac{x}{2}\right) = 2\cos\left(\frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)\cos\left(\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right). \tag{3}$$

Since $\cos(\pi/4 - x/2)$ is the last factor of (2), we can substitute (3) into (2) to obtain

$$\cos x = 2^2\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{\pi}{8} - \frac{x}{4}\right)\cos\left(\frac{\pi}{4} - \frac{\pi}{8} + \frac{x}{4}\right). \tag{4}$$

Now the last factor is $\cos(\pi/4 - \pi/8 + x/4)$. We get an expression for this by applying (2) again with $(\pi/4 - \pi/8 + x/4)$ in place of x,

$$\begin{aligned}\cos&\left(\frac{\pi}{4} - \frac{\pi}{8} + \frac{x}{4}\right)\\ &= 2\cos\left(\frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{8} + \frac{x}{4}\right)\right)\cos\left(\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{8} + \frac{x}{4}\right)\right),\end{aligned} \tag{5}$$

and now, the substitution of (5) into (4) yields

$$\begin{aligned}\cos x = 2^3&\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{\pi}{8} - \frac{x}{4}\right)\cos\left(\frac{\pi}{4} + \frac{\pi}{8} - \frac{\pi}{16} + \frac{x}{8}\right)\\ &\cdot\cos\left(\frac{\pi}{4} - \frac{\pi}{8} + \frac{\pi}{16} - \frac{x}{8}\right).\end{aligned}$$

If we proceed in like manner, choosing in each step the last factor $\cos(\pi/4 - \cdot/2)$ for the next iteration, it is easy to verify that for each positive integer k,

$$\begin{aligned}\cos x = 2^k&\left(\prod_{j=0}^{k-1}\cos\left(\frac{\pi}{4} + \sum_{i=1}^{j}(-1)^{i+1}\frac{\pi}{2^{i+2}} + (-1)^{j+2}\frac{x}{2^{j+1}}\right)\right)\\ &\cdot\cos\left(\sum_{i=1}^{k}(-1)^{i+1}\frac{\pi}{2^{i+1}} + (-1)^k\frac{x}{2^k}\right),\end{aligned} \tag{6}$$

with the understanding that the empty sum $\sum_{i=1}^{0}$ equals 0.

Let a_j denote the jth Jacobsthal number, recursively defined by $a_0 = 0$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + 2a_n$ for $n \geq 0$ [**9**]. Then a_j can be written in closed form as $a_j = (2^j - (-1)^j)/3$. These numbers satisfy the identities $2^j + a_j = a_{j+2}$ (for $j \geq 0$) and $\lim_{j\to\infty}\left(a_j/2^{j+1}\right) = 1/6$. With this notation in mind, our first result reads:

THEOREM 1. *For any real x,*

$$\frac{2}{\sqrt{3}}\cos x = \prod_{n=0}^{\infty} 2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}} + \frac{(-1)^{n+2}x}{2^{n+1}}\right). \tag{7}$$

Proof. We invite the reader to evaluate the geometric series, and in that way to verify that for all nonnegative integers j,

$$\frac{\pi}{4} + \sum_{i=1}^{j}(-1)^{i+1}\frac{\pi}{2^{i+2}} = \frac{\pi}{4} + \frac{(2^j - (-1)^j)\pi}{3\cdot 2^{j+2}} = \frac{\pi}{4} + \frac{a_j\pi}{2^{j+2}} = \frac{a_{j+2}\pi}{2^{j+2}}, \tag{8}$$

$$\sum_{i=1}^{k}(-1)^{i+1}\frac{\pi}{2^{i+1}} = 2\frac{(2^k - (-1)^k)\pi}{3\cdot 2^{k+2}} = \frac{a_k\pi}{2^{k+1}}. \tag{9}$$

Substituting (8) and (9) into (6) we find that for each real x and every positive integer k,

$$\frac{\cos x}{\cos\left(\frac{a_k\pi}{2^{k+1}}+\frac{(-1)^k x}{2^k}\right)}=\prod_{j=0}^{k-1} 2\cos\left(\frac{a_{j+2}\pi}{2^{j+2}}+\frac{(-1)^{j+2}x}{2^{j+1}}\right).$$

Finally, formula (7) is a consequence of the fact that, for arbitrary fixed x,

$$\begin{aligned}\lim_{k\to\infty}\cos\left(\frac{a_k\pi}{2^{k+1}}+\frac{(-1)^k x}{2^k}\right)&=\cos\left(\lim_{k\to\infty}\left(\frac{a_k\pi}{2^{k+1}}+\frac{(-1)^k x}{2^k}\right)\right)\\&=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}.\end{aligned}$$ ■

The formula in Theorem 1 is the infinite product of cosines that we declared as our first objective.

Servi's formula

Our purpose now is to rewrite Theorem 1 in terms of nested square roots of 2. The key tool for this purpose is Servi's formula.

SERVI'S FORMULA [**7**]. *Fix n a positive integer. If $b_0 \in [-2, 2]$ and $b_j \in \{-1, 1\}$ for $j = 1, 2, \ldots, n$, then*

$$\begin{aligned}&b_n\sqrt{2+b_{n-1}\sqrt{2+b_{n-2}\sqrt{2+\cdots+b_1\sqrt{2+2\sin(b_0\pi/4)}}}}\\&=2\cos\left(\left(\frac{1}{2}-\frac{b_n}{2^2}-\frac{b_nb_{n-1}}{2^3}-\cdots-\frac{b_nb_{n-1}\cdots b_1b_0}{2^{n+2}}\right)\pi\right)\\&=2\cos\left(\left(2^{-1}-\sum_{j=1}^{n+1}\left(2^{-(j+1)}\prod_{i=0}^{j-1}b_{n-i}\right)\right)\pi\right).\end{aligned}\tag{10}$$

Thus, to achieve our aim, if for each n we find values $b_0 = b_0(n)$, $b_1 = b_1(n), \ldots,$ $b_n = b_n(n)$ with the restrictions above and such that

$$\frac{a_{n+2}\pi}{2^{n+2}}+\frac{(-1)^{n+2}x}{2^{n+1}}=\left(2^{-1}-\sum_{j=1}^{n+1}\left(2^{-(j+1)}\prod_{i=0}^{j-1}b_{n-i}\right)\right)\pi,\tag{11}$$

then we can use Servi's formula (10) to transform the infinite product of (twice the) cosines (7) into an infinite product of nested square roots, obtaining one of the Viète-like formulae. We begin with the simple case corresponding to the choice $x = 0$ in (7) which, as noted above, is due to A. Levin [**2**].

THEOREM 2.

$$\frac{2}{\sqrt{3}}=\sqrt{2}\sqrt{2-\sqrt{2}}\sqrt{2-\sqrt{2-\sqrt{2}}}\cdots.\tag{12}$$

Proof. First we choose $x = 0$ in (7) to get

$$\frac{2}{\sqrt{3}} = \prod_{n=0}^{\infty} 2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}}\right). \tag{13}$$

Second, we evaluate the geometric series $\sum_{j=0}^{n+1}\left((-1)^j/2^{j+1}\right)$ to obtain

$$\begin{aligned} 2^{-1} - \sum_{j=1}^{n+1} 2^{-(j+1)}(-1)^{j-1} &= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots + (-1)^{n+1}\frac{1}{2^{n+2}} \\ &= \frac{2^{n+2} - (-1)^{n+2}}{3 \cdot 2^{n+2}} = \frac{a_{n+2}}{2^{n+2}}. \end{aligned} \tag{14}$$

Thus, comparing (14) and (11), the latter formula with the value $x = 0$, for each n we are left with the task of determining the coefficients $b_0, b_1, \ldots, b_n$ such that

$$\prod_{i=0}^{j-1} b_{n-i} = (-1)^{j-1} \quad \text{for} \quad j = 1, 2, \ldots, n+1.$$

Clearly, the values $b_n = 1$, $b_{n-1} = b_{n-2} = \cdots = b_0 = -1$ satisfy the required condition. Therefore, we conclude from (14) and (10) that

$$2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}}\right) = \underbrace{\sqrt{2 - \sqrt{2 - \sqrt{2 - \cdots - \sqrt{2}}}}}_{n+1 \text{ square roots}}, \qquad n = 0, 1, 2, \ldots. \tag{15}$$

(The case $n = 0$ corresponds to the direct evaluation $2\cos(a_2\pi/4) = 2\cos(\pi/4) = \sqrt{2}$.) Finally, the replacement of (15) into (13) establishes (12). ■

Without imposing $x = 0$ in (7), the following generalization of Theorem 2 holds:

THEOREM 3. *Let $z \in [-2, 2]$. Then*

$$\sqrt{\frac{4 - z^2}{3}} = \sqrt{2 - z}\sqrt{2 - \sqrt{2 - z}}\sqrt{2 - \sqrt{2 - \sqrt{2 - z}}} \cdots .$$

Proof. For fixed $z \in [-2, 2]$, and for a nonnegative integer n, let the coefficients b_j be defined by means of

$$b_0 = -\frac{4}{\pi}\arcsin\frac{z}{2}, \qquad b_1 = b_2 = \cdots = b_n = -1, \qquad b_{n+1} = 1,$$

in the case that $n > 0$, and

$$b_0 = -\frac{4}{\pi}\arcsin\frac{z}{2}, \qquad b_1 = 1,$$

when $n = 0$. Then, Servi's formula (10) transforms to

$$b_{n+1}\sqrt{2+b_n\sqrt{2+\cdots+b_1\sqrt{2+2\sin b_0\pi/4}}} = \underbrace{\sqrt{2-\sqrt{2-\cdots-\sqrt{2-z}}}}_{n+1 \text{ square roots}}$$

$$= 2\cos\left(\left(\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\cdots+(-1)^{n+1}\frac{1}{2^{n+2}}+(-1)^{n+2}\frac{4\arcsin(z/2)}{2^{n+3}\pi}\right)\pi\right)$$

$$= 2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}}+(-1)^{n+2}\frac{\arcsin(z/2)}{2^{n+1}}\right) = 2\cos\left(\frac{a_{n+2}\pi}{2^{n+2}}+\frac{(-1)^{n+2}x}{2^{n+1}}\right),$$

where in the last equality we have defined $x = \arcsin(z/2)$. Thus, the substitutions of the equation above, together with the identity $\cos x = \cos(\arcsin(z/2)) = \sqrt{1-(z^2/4)}$, into formula (7), give us the desired conclusion. ∎

Theorem 3 is not new. To the best of our knowledge, like Theorem 2 it is due to A. Levin [**2**, formula (48)]. A later paper by Levin [**3**] is also relevant as background.

Obtaining the original Viete's formula

In this section we obtain the original Viète's formula using the technique developed above. For convenience, we define the *plus* and *minus* functions by

$$p(x) = \frac{\pi}{4}+\frac{x}{2} \qquad \text{and} \qquad m(x) = \frac{\pi}{4}-\frac{x}{2}.$$

We also recursively define the iterations of p by $p^0(x) = x$, $p^1(x) = p(x)$, $p^2(x) = p(p(x))$, and in general $p^{k+1}(x) = p(p^k(x))$, and similarly the iterations of m by $m^0(x) = x$ and $m^{k+1}(x) = m(m^k(x))$.

With this notation, equation (2) can be rewritten as

$$\cos x = 2\cos(p(x))\cos(m(x)). \tag{16}$$

(i) Iterating (16) k times, by always running in each iteration the cosine factor associated with the plus function, we get the analog of (6):

$$\begin{aligned}\cos x &= 2\cos(m(x))\cdot\cos(p(x))\\ &= 2^2\cos(m(x))\cos(m(p(x)))\cdot\cos(p^2(x))\\ &= 2^3\cos(m(x))\cos(m(p(x)))\cos(m(p^2(x)))\cdot\cos(p^3(x))\\ &\ \vdots\\ &= 2^k\left(\prod_{j=0}^{k-1}\cos(m(p^j(x)))\right)\cdot\cos(p^k(x)).\end{aligned} \tag{17}$$

(ii) Now, for each nonnegative integer j, we give explicit expressions for both $p^j(x)$ and $m(p^j(x))$:

$$p^j(x) = \frac{(2^j-1)\pi}{2^{j+1}}+\frac{x}{2^j}, \tag{18}$$

$$m(p^j(x)) = \frac{\pi}{2^{j+2}}-\frac{x}{2^{j+1}}. \tag{19}$$

We use induction to prove our first assertion. First, since p^0 is the identity map, it is clear that (18) holds for $j = 0$. Now suppose that (18) holds for $j = k$. Then,

$$\begin{aligned} p^{k+1}(x) &= \frac{\pi}{4} + \frac{p^k(x)}{2} = \frac{\pi}{4} + \frac{1}{2}\left(\frac{(2^k-1)\pi}{2^{k+1}} + \frac{x}{2^k}\right) \\ &= (2^k + 2^k - 1)\frac{\pi}{2^{k+2}} + \frac{x}{2^{k+1}} \\ &= \frac{(2^{k+1}-1)\pi}{2^{k+2}} + \frac{x}{2^{k+1}}, \end{aligned}$$

and this concludes the first part. For the second assertion (19), note that

$$m(p^j(x)) = \frac{\pi}{4} - \frac{1}{2}\left(\frac{(2^j-1)\pi}{2^{j+1}} + \frac{x}{2^j}\right) = \frac{\pi}{2^{j+2}} - \frac{x}{2^{j+1}},$$

as stated above.

(iii) Replacing (18) and (19) in (17), we obtain

$$\cos x = \left(\prod_{j=0}^{k-1} 2\cos\left(\frac{\pi}{2^{j+2}} - \frac{x}{2^{j+1}}\right)\right)\cos\left(\frac{(2^k-1)\pi}{2^{k+1}} + \frac{x}{2^k}\right).$$

(iv) Defining a new variable y by means of $x = (\pi/2) - y$, we have

$$\begin{aligned} \sin y &= \cos\left(\frac{\pi}{2} - y\right) \\ &= \left(\prod_{j=0}^{k-1} 2\cos\left(\frac{\pi}{2^{j+2}} - \frac{\frac{\pi}{2}-y}{2^{j+1}}\right)\right)\cos\left(\frac{(2^k-1)\pi}{2^{k+1}} + \frac{\frac{\pi}{2}-y}{2^k}\right) \\ &= \left(\prod_{j=0}^{k-1} 2\cos\left(\frac{y}{2^{j+1}}\right)\right)\cos\left(\frac{\pi}{2} - \frac{y}{2^k}\right) \\ &= 2^k\left(\prod_{j=0}^{k-1}\cos\left(\frac{y}{2^{j+1}}\right)\right)\sin\left(\frac{y}{2^k}\right), \end{aligned}$$

and the choice $y = \pi/2$ gives us

$$\frac{2}{\pi}\frac{\pi/2^{k+1}}{\sin(\pi/2^{k+1})} = \prod_{j=2}^{k+1}\cos\frac{\pi}{2^j},$$

and this is the point where the classical proof (starting with the double angle formula) and the derivation coming from the trigonometric relation (2) converge (see [**8**, pp. 252–253] for further details).

New Viète-like formulae

Now we iterate (16) by running, in the first step, the cosine factor associated with the minus function; in the second step, the cosine factor associated with the plus function (this completes a cycle); in the third step, the cosine factor associated with the minus function; in the fourth step, the cosine factor associated with the plus function (this completes another cycle), ... and so on. With this in mind, and using that $m(m(x)) = \pi/4 - (1/2)(\pi/4 - x/2) = (1/2)(\pi/4 + x/2) = p(x)/2$, we have

$$\begin{aligned}\cos x &= \big(2\cos(p(x))\big)\cos(m(x)) \\ &= \big(2^2\cos(p(x))\cos(m(m(x)))\big)\cos(p(m(x))) \\ &= \left(2^2\cos(p(x))\cos\left(\frac{p}{2}(x)\right)\right)\cos(p(m(x))) \\ &= \left(2^3\cos(p(x))\cos\left(\frac{p}{2}(x)\right)\cos(p(p(m(x))))\right)\cos(m(p(m(x)))) \\ &= \left(2^4\cos(p(x))\cos\left(\frac{p}{2}(x)\right)\cos(p(p(m(x))))\cos\left(\frac{p}{2}(p(m(x)))\right)\right) \\ &\quad\cdot\cos(p(m(p(m(x))))) \\ &= \cdots.\end{aligned}$$

In the sequel, and for notational purposes, for a fixed real number x and a fixed non-negative integer j we will write:

$$\begin{aligned}[s, (p,m)^j] &= \cos(s((p,m)^j(x))) \\ &= \cos(s(\underbrace{\underbrace{p(m}\underbrace{(p(m}\cdots\underbrace{(p(m}}_{j\text{ times}}(x)))\cdots))))),\end{aligned}$$

where s stands for one of the signs functions p, m, or $p/2$. With this notation, the iteration described above can be rewritten as:

$$\begin{aligned}\cos x &= 2[p][m] = 2^2[p]\left[\frac{p}{2}\right][(p,m)] = 2^3[p]\left[\frac{p}{2}\right][p,(p,m)][m,(p,m)] \\ &= 2^4[p]\left[\frac{p}{2}\right][p,(p,m)]\left[\frac{p}{2},(p,m)\right][(p,m)^2] \\ &= 2^5[p]\left[\frac{p}{2}\right][p,(p,m)]\left[\frac{p}{2},(p,m)\right][p,(p,m)^2][m,(p,m)^2] \\ &= 2^6[p]\left[\frac{p}{2}\right][p,(p,m)]\left[\frac{p}{2},(p,m)\right][p,(p,m)^2]\left[\frac{p}{2},(p,m)^2\right][(p,m)^3] \\ &= \cdots.\end{aligned}$$

Therefore, after making an odd number of iterations, we get

$$\cos x = 2^{2k}\left(\prod_{j=0}^{k-1}\left([p,(p,m)^j]\left[\frac{p}{2},(p,m)^j\right]\right)\right)\left(2[p,(p,m)^k][m,(p,m)^k]\right), \tag{20}$$

with the understanding that the empty product $\prod_{j=0}^{-1}$ equals 1, and where k is the integer part of half the number of iterations ($k = 0, 1, 2, \ldots$). In like manner, when we make an even number of iterations, we obtain

$$\cos x = 2^{2k}\left(\prod_{j=0}^{k-1}\left([p,(p,m)^j]\left[\frac{p}{2},(p,m)^j\right]\right)\right)[(p,m)^k], \tag{21}$$

here being k half the number of iterations ($k = 1, 2, \ldots$).

As in item (ii) in the derivation of the original Viète's formula, we now give explicit expressions for $(p,m)^j(x)$, $p((p,m)^j(x))$ and $m((p,m)^j(x))$.

PROPOSITION 1. *For each nonnegative integer j,*

$$(p,m)^j(x) = \frac{3(4^j-(-1)^j)}{10}\frac{\pi}{4^j} + \frac{(-1)^j x}{4^j}, \tag{22}$$

$$p((p,m)^j(x)) = \frac{2\cdot 4^{j+1}+3(-1)^{j+1}}{5}\frac{\pi}{4^{j+1}} + \frac{(-1)^j 2x}{4^{j+1}}, \tag{23}$$

$$m((p,m)^j(x)) = \frac{2\cdot 4^{j}+3(-1)^{j}}{5}\frac{\pi}{4^{j+1}} - \frac{(-1)^j 2x}{4^{j+1}}. \tag{24}$$

Proof. Use induction to prove (22). After that, assertions (23) and (24) can be verified by direct evaluations. ∎

Our second (and, to the best of our knowledge, new) infinite product reads:

THEOREM 4. *Set $d_j = (2\cdot 4^j + 3(-1)^j)/5$ for each nonnegative integer j. Then, for each real number x,*

$$\frac{2\sqrt{2}\cos x}{\sqrt{5-\sqrt{5}}} = \prod_{n=0}^{\infty}\left(2\cos\left(\frac{d_{n+1}\pi}{4^{n+1}} + \frac{(-1)^n 2x}{4^{n+1}}\right)2\cos\left(\frac{d_{n+1}\pi}{2\cdot 4^{n+1}} + \frac{(-1)^n x}{4^{n+1}}\right)\right).$$

Proof. We only have to take limits as k tends to infinity in (20) and (21), taking into account that Proposition 1 yields

$$\begin{aligned}\lim_{k\to\infty}\left(2[p,(p,m)^k][m,(p,m)^k]\right) &= 2\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{\pi}{10}\right)\\ &= \cos\left(\frac{2\pi}{5}+\frac{\pi}{10}\right)+\cos\left(\frac{2\pi}{5}-\frac{\pi}{10}\right)\\ &= \cos\left(\frac{3\pi}{10}\right) = \lim_{k\to\infty}\left([(p,m)^k]\right),\end{aligned}$$

where $\cos(3\pi/10) = \sqrt{5-\sqrt{5}}/(2\sqrt{2})$. ∎

In order to get a Viète-like formula, using Servi's formula, we need to know how the numbers d_n are connected with sums or differences of the numbers $1/2$, $1/4$, $1/8$, $\ldots$, $1/2^{2n+1}$. To this end, we give the following technical result.

LEMMA. *For each nonnegative integer n,*

$$\frac{d_{n+1}}{4^{n+1}} = \frac{1}{2} - \sum_{j=2}^{2n+2}\frac{(-1)^{\lfloor\frac{j-1}{2}\rfloor}}{2^j}, \tag{25}$$

$$\frac{d_{n+1}}{2\cdot 4^{n+1}} = \frac{1}{2} - \sum_{j=2}^{2n+3}\frac{(-1)^{\lfloor\frac{j-2}{2}\rfloor}}{2^j}. \tag{26}$$

We leave the proof of the lemma to the reader (once again, it is an inductive argument). Now we are in position to establish a completely new (to the best of our knowledge) Viète-like infinite product of nested square roots, in which each factor is in turn the product of two nested radicals.

(26), we obtain

$$b_{2n+2}\sqrt{2+b_{2n+1}\sqrt{2+\cdots+b_1\sqrt{2+2\sin b_0\pi/4}}}$$

$$=\underbrace{\sqrt{2+\sqrt{2-\sqrt{2+\cdots\sqrt{2-\sqrt{2+\sqrt{2-z}}}}}}}_{2n+2\text{ square roots}}$$

$$=2\cos\left(\left(\frac{1}{2}-\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}-\frac{1}{2^6}-\frac{1}{2^7}\right.\right.$$

$$\left.\left.-\cdots-(-1)^n\frac{1}{2^{2n+2}}-(-1)^n\frac{1}{2^{2n+3}}-(-1)^n\frac{-4\arcsin(z/2)}{2^{2n+4}\pi}\right)\pi\right)$$

$$=2\cos\left(\left(\frac{1}{2}-\sum_{j=2}^{2n+3}\frac{(-1)^{\lfloor\frac{j-2}{2}\rfloor}}{2^j}+\frac{(-1)^n x}{2^{2n+2}\pi}\right)\pi\right)$$

$$=2\cos\left(\frac{d_{n+1}\pi}{2\cdot 4^{n+1}}+\frac{(-1)^n x}{4^{n+1}}\right). \tag{28}$$

In conclusion, if we replace (27) and (28) in the formula of Theorem 4, and we use that $\cos x=\sqrt{1-(z^2/4)}$, we get the desired result. ■

Concluding remarks

We hope that our approach will be helpful for those interested readers who want to obtain their own Viète-like formulae. They should consider their own particular way for running the iterations of the key formula (16). For instance, interchanging the roles of the plus and minus functions in the procedure described above, we get that $\sqrt{2(4-z^2)}/\sqrt{5+\sqrt{5}}$ equals

$$\left(\sqrt{2+z}\sqrt{2-\sqrt{2+z}}\right)\left(\sqrt{2+\sqrt{2-\sqrt{2+z}}}\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+z}}}}\right)\cdots.$$

for each z in $[-2,2]$. (By the way, as addressed in [**5**], the formula above yields a nice representation of the Golden Section $\Phi=(1+\sqrt{5})/2$ using only the number 2.) Moreover, we are currently in the process of investigating the possibility of obtaining a general Viète-like formulae that may describe all the cases coming from the different iterative schemes acting over (16). To this problem, and other related ones, we hope to return elsewhere [**6**].

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Summary In this article, the authors show that Viète's formula is only the tip of the iceberg. Under the surface, they search for curious and interesting Viète-like infinite products, rare species made of products of nested square roots of 2, but here with some minus signs occurring inside. To explore this fascinating world, they only use the simple trigonometric identity $\cos x = 2\cos((\pi + 2x)/4)\cos((\pi - 2x)/4)$, combined with a recent formula by L. D. Servi.

Sitting Down for Dinner at a Twin Convention

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Although I know of the existence of twin conventions, attended solely by n sets of identical twins for some positive integer n, I must confess to having absolutely no idea of what actually goes on at these gatherings. Still, I know that such events would give a budding combinatorialist the perfect opportunity to hone his or her skills.

Suppose, for example, that a photographer is arranging all the twins in a line in order to take the official convention photograph. Then, making the assumption that identical twins are visually indistinguishable from one another, and hence, for the sake of the photograph, that it would be pointless to interchange the positions of any such pair, our combinatorialist might like to calculate the number of possible arrangements. They would soon realize, however, that this is a relatively straightforward calculation. No doubt the answer

$$\frac{(2n)!}{2^n}$$

would be obtained almost immediately, and a more demanding problem sought.

Indeed, the purpose of this article is to discuss, and then show how to solve, a somewhat stiffer twin-related combinatorial challenge.

Suppose that the n sets of twins are taking their seats for dinner, and that there is at least one person sitting at each of the k tables (this clearly requires $2n \geq k$). By way of an illustration, FIGURE 1 depicts one of many possible ways in which eight sets of twins may be seated at five tables. We might now ask this question:

> In how many ways can these n sets of twins be distributed among the tables, if exactly k tables are occupied?

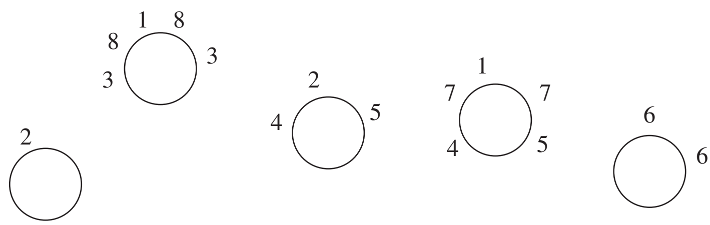

Figure 1 Five tables occupied by eight sets of twins

In this article we will never be concerned with the order of persons around a table, or with the order of the tables. Nor will we ever count empty tables. We care only who sits with whom. Our challenge, then, is to count the number of different ways in which the $2n$ attendees can be partitioned into k nonempty groups (or tables).

As we will see, there are in fact several variations of the problem to consider. But first, we briefly take a look at the corresponding situation for n distinguishable people.

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What if they are not twins? We assume, as always, that the way individuals are positioned at a particular table is irrelevant, and that the k tables are indistinguishable from one another. If there are n distinguishable people, then the *Stirling number of the second kind* $S(n,k)$ counts the number of ways in which these people may be seated. In other words, $S(n,k)$ enumerates the partitions of the set $\{1,2,\ldots,n\}$ into exactly k nonempty parts. We say that $S(0,0)=1$, and that $S(0,k)=S(n,0)=0$ for all positive integers k and n, by definition. Then these numbers satisfy the recurrence relation

$$S(n,k)=kS(n-1,k)+S(n-1,k-1), \tag{1}$$

for all positive integers k and n, as we now explain. If the part $\{n\}$ is added to any partition of $\{1,2,\ldots,n-1\}$ into $k-1$ parts, then we obtain a partition of $\{1,2,\ldots,n\}$ into k parts. Also, given any partition of $\{1,2,\ldots,n-1\}$ into k parts we may, by inserting the element n into one these parts, obtain a partition of $\{1,2,\ldots,n\}$ into k parts. Thus, each partition of $\{1,2,\ldots,n-1\}$ into k parts gives rise to k partitions of $\{1,2,\ldots,n\}$ into k parts. Furthermore, each of the partitions of $\{1,2,\ldots,n\}$ into k parts arises exactly once in these constructions.

The numbers $S(n,k)$ lead to a triangular table, which can be seen in [**12**] and also in the *On-line Encyclopedia of Integer Sequences* [**16**] as sequence A008277. Many well-known results are associated with the Stirling numbers of the second kind; see [**6, 7, 8, 9, 11, 12**], for example.

Refining the question

Let us now return to the case of n pairs of identical twins sitting at k tables. If we decided to treat the twins as distinguishable, then we would have $2n$ distinguishable people at our convention, and the answer to our question would be $S(2n,k)$. We want the problem to be somewhat more challenging than that, so in this article we will never again treat identical twins as distinguishable. That is, when we count ways to arrange the attendees, two plans that differ only in that some people are exchanged with their twins count for us as just one arrangement.

We will show how to enumerate the ways of seating n sets of identical twins at k indistinguishable tables, subject to the proviso that each table is occupied. This will require the use of *multisets*, which are defined in the following section. As we will see, the fact that there are now repeated elements means that we may perform the partitioning in several different ways, according to what, if any, restrictions are placed on repeated elements in a part, or indeed on repeated parts in a partition.

A brief history of multisets

Multisets generalize the notion of a set in order to cater for the possibility of repeated elements, as in the scenario involving twins mentioned above. The following formal definition may also be found in [**1, 10, 11, 17**]. We use $\mathbb{N}$ for $\{1,2,\ldots\}$.

DEFINITION. A multiset is a pair (X,m), where X is some set and m is a function $m : X \mapsto \mathbb{N}$. The set X is called the set of underlying elements. For each $x \in X$, the *multiplicity* of x is given by $m(x)$. A multiset is called an n-multiset if $\sum_{x\in X} m(x)=n$ for some $n \in \mathbb{N}$.

In unions of multisets, the multiplicities are compounded. Thus, for example,

$$\{1, 1, 2, 3, 3, 4, 4, 4\} \cup \{1, 2, 2, 2, 5, 5\} = \{1, 1, 1, 2, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5\}.$$

Curious readers may find an algorithm, though not a formula, in [**13**] for generating all possible partitions of any given multiset.

The term *multiset* appears first to have been used in a private correspondence between N. G. de Bruijn and D. E. Knuth [**5**], and is now in common usage, having replaced terms such as 'bag', 'heap', and 'occurrence set'. However, despite the name being relatively modern, the concept most certainly is not. The ancient system of using tally marks to count a collection of objects, such as livestock, might be regarded as the first, albeit unwitting, application of multisets. Thus, for example, $||||$ could have been used to denote that a person owned four sheep. In this sense, the notion of a multiset is intertwined with the very origin of numbers.

In [**2**] there is a reference to work carried out in the sixteenth century by Marius Nizolius, in which the concept of a multiset arises. Of course, it needs to be borne in mind that the standard definition of a set, let alone that of a multiset, was not actually formulated until the end of the nineteenth century (by Georg Cantor). At around this time, Richard Dedekind, while working on aspects of number, used the notion of a multiset in connection with functions. He noted that any element in the range of a function may be regarded as occurring with a multiplicity equal to the number of pre-images it possesses. The author of [**5**] suggests that the first serious mathematical treatment of multisets was undertaken by Hassler Whitney in the early 1930s, while he was working on characteristic functions of sets.

From a combinatorial point of view, two important papers on multisets were published by Edward Bender [**3, 4**] in 1974 and 1984, respectively. The first of these is related to our work here, although we employ different methods and none of the generating functions or formulas we give in the present paper appear there. Two more recent papers, [**10, 11**], consider various enumeration problems for a multiset comprising a group of n people containing exactly one subgroup of identical r-tuplets.

We have, of course, only scratched the surface of this vast topic, and interested readers might like to refer to [**5, 15**] for a considerably more detailed account of the history of multisets.

Some definitions and examples

DEFINITION. Let $\mathcal{M}(n, j)$ denote the nj-multiset in which each of the elements $1, 2, 3, \ldots, n$ has multiplicity j.

We are concerned here with $\mathcal{M}(n, 2) = \{1, 1, 2, 2, 3, 3, \ldots, n, n\}$, although we shall briefly mention $\mathcal{M}(n, 3)$ later in the paper. Our problem corresponds to enumerating the partitions of $\mathcal{M}(n, 2)$ into exactly k nonempty parts. As we have said, there is more than one way of performing this enumeration, depending on whether we allow repeated parts or repeated elements within a part. We identify four possibilities, as summarized in the following definition.

DEFINITION. Let n and k be nonnegative integers.

(i) $A(n, k)$ enumerates the partitions of $\mathcal{M}(n, 2)$ into exactly k nonempty parts.

(ii) $B(n, k)$ enumerates the partitions of $\mathcal{M}(n, 2)$ into exactly k nonempty parts such that all these parts are distinct.

(iii) $C(n,k)$ enumerates the partitions of $\mathcal{M}(n,2)$ into exactly k nonempty parts such that none of them contains a pair of identical elements.

(iv) $D(n,k)$ enumerates the partitions of $\mathcal{M}(n,2)$ into exactly k nonempty parts such that all these parts are distinct and none of them contains a pair of identical elements.

We set $A(0,0)=1$ by definition. Note also that $A(n,0)=A(0,k)=0$ for any $n,k\in\mathbb{N}$, and similarly for $B(n,k)$, $C(n,k)$ and $D(n,k)$. Furthermore, as is easily verified, $A(n,1)=B(n,1)=C(n,2)=1$ and $C(n,1)=D(n,1)=D(n,2)=0$ for all $n\in\mathbb{N}$. It is worth pointing out here that the partition depicted in FIGURE 1 contributes to both $A(8,5)$ and $B(8,5)$, but not to $C(8,5)$ or $D(8,5)$.

For a given value of k, the smallest values of n for which $A(n,k)$ and $B(n,k)$ are nonzero are given by

$$\left\lfloor \frac{k+1}{2} \right\rfloor \qquad \text{and} \qquad \left\lfloor \frac{2(k+1)}{3} \right\rfloor,$$

respectively. These properties can be seen quite clearly in TABLES 1 and 2. The numbers $S(n,k)$ lead to a triangular table. We see from the tables that the same cannot quite be said for each of $A(n,k)$ and $B(n,k)$; and neither can it be said for $C(n,k)$ or $D(n,k)$. We shall, however, take the liberty of referring to these as "number triangles."

TABLE 1: The number of partitions $A(n,k)$ of $\mathcal{M}(n,2)$ into exactly k nonempty parts

n	$A(n,1)$	$A(n,2)$	$A(n,3)$	$A(n,4)$	$A(n,5)$	$A(n,6)$
1	1	1				
2	1	4	3	1		
3	1	13	26	19	6	1
4	1	40	183	259	163	55
5	1	121	1190	3115	3373	1884
6	1	364	7443	34891	62240	54522
7	1	1093	45626	374059	1072316	1429883
8	1	3280	276783	3903019	17656653	35165417

TABLE 2: The number of partitions $B(n,k)$ of $\mathcal{M}(n,2)$ into exactly k nonempty parts such that all these parts are distinct

n	$B(n,1)$	$B(n,2)$	$B(n,3)$	$B(n,4)$	$B(n,5)$	$B(n,6)$
1	1					
2	1	3	1			
3	1	12	20	7		
4	1	39	169	186	59	3
5	1	120	1160	2755	2243	661
6	1	363	7381	33270	52060	33604
7	1	1092	45500	367087	988750	1126874
8	1	3279	276529	3873786	17005149	31177517

As practice, let us evaluate $A(3,3)$, $B(3,3)$, $C(3,3)$, and $D(3,3)$. First, the list below gives all partitions of $\mathcal{M}(3,2)$ into exactly three parts such that all parts are distinct:

$$\begin{array}{lll}
\{\{1\},\{2\},\{1,2,3,3\}\} & \{\{1\},\{3\},\{1,2,2,3\}\} & \{\{2\},\{3\},\{1,1,2,3\}\} \\
\{\{1\},\{2,2\},\{1,3,3\}\} & \{\{1\},\{3,3\},\{1,2,2\}\} & \{\{2\},\{1,1\},\{2,3,4\}\} \\
\{\{2\},\{3,3\},\{1,1,2\}\} & \{\{3\},\{1,1\},\{2,2,3\}\} & \{\{3\},\{2,2\},\{1,1,3\}\} \\
\{\{1\},\{2,3\},\{1,2,3\}\} & \{\{2\},\{1,3\},\{1,2,3\}\} & \{\{3\},\{1,2\},\{1,2,3\}\} \\
\{\{1\},\{1,2\},\{2,3,3\}\} & \{\{1\},\{1,3\},\{2,2,3\}\} & \{\{2\},\{1,2\},\{1,3,3\}\} \\
\{\{2\},\{2,3\},\{1,1,3\}\} & \{\{3\},\{1,3\},\{1,2,2\}\} & \{\{3\},\{2,3\},\{1,1,2\}\} \\
\{\{1,1\},\{2,2\},\{3,3\}\} & \{\{1,2\},\{1,3\},\{2,3\}\} &
\end{array}$$

This shows that $B(3,3) = 20$. Second, listing all partitions containing repeated parts,

$$\begin{array}{lll}
\{\{1\},\{1\},\{2,2,3,3\}\} & \{\{2\},\{2\},\{1,1,3,3\}\} & \{\{3\},\{3\},\{1,1,2,2\}\} \\
\{\{1,2\},\{1,2\},\{3,3\}\} & \{\{1,3\},\{1,3\},\{2,2\}\} & \{\{2,3\},\{2,3\},\{1,1\}\},
\end{array}$$

we see that $A(3,3) = B(3,3) + 6 = 26$. Third, the partitions of $\mathcal{M}(3,2)$ into exactly three parts, such that all these parts are distinct and none of them contains a pair of identical elements, are given by

$$\begin{array}{ll}
\{\{1\},\{2,3\},\{1,2,3\}\} & \{\{2\},\{1,3\},\{1,2,3\}\} \\
\{\{3\},\{1,2\},\{1,2,3\}\} & \{\{1,2\},\{1,3\},\{2,3\}\},
\end{array}$$

showing that $D(3,3) = 4$. Last, note in this particular case that $C(3,3) = 4$ also, although it is not generally true that $C(n,k) = D(n,k)$.

Counting factorizations Finally in this section, we give an alternative combinatorial interpretation of $A(n,k)$. Let $N = (p_1 p_2 p_3 \cdots p_n)^2$, where $p_1, p_2, p_3, \ldots, p_n$ are distinct primes. Then, disregarding the order of the factors, $A(n,k)$ gives the number of ways in which N can be expressed as a product of k positive integers, each of which is greater than 1. In the same context, $B(n,k)$ gives the number of ways in which N can be expressed as a product of k *distinct* positive integers, each of which is greater than 1; $C(n,k)$ and $D(n,k)$ count corresponding products with the requirement that each factor be squarefree.

The number triangle $B(n, k)$

We now embark on a quest to obtain, in a recursive manner, formulas that will allow us to evaluate $B(n,k)$. In particular, we shall obtain $B(8,5)$, which relates to the situation depicted in FIGURE 1. The reason for tackling $B(n,k)$ first is that, as we will see below, it is fairly straightforward to obtain the number triangle for $A(n,k)$ from that of $B(n,k)$.

While the Stirling numbers of the second kind satisfy the recurrence relation given in (1), the corresponding situation for $B(n,k)$ is a little more complicated. Indeed, it satisfies an extended recurrence relation, as is shown below. In this section, we show how to obtain recursively a formula for the nth term in column k of the number triangle $B(n,k)$.

We define sums of the form $\sum_{j=a}^{b} f(j)$ to be zero when $a > b$. Note that $B(n,k)$ is zero when k is negative.

THEOREM 1. *For $n \geq 0$,*

$$B(n+1,k) = \binom{k+1}{2} B(n,k) + kB(n,k-1) + p(n,k) + q(n,k),$$

where

$$p(n,k) = \sum_{j=1}^{n} \binom{n}{j} \{(k-1)B(n-j,k-2) + B(n-j,k-3)\},$$

$$q(n,k) = \sum_{j=2}^{n} \binom{n}{j} S(j,2) B(n-j,k-4),$$

and $B(0,0) = 1$ by definition.

Proof. Let $\mathcal{U}(n,k)$ denote the set of all partitions of $\mathcal{M}(n,2)$ into exactly k nonempty distinct parts. To each $u \in \mathcal{U}(n,k-1)$, we may append the part $\{n+1, n+1\}$ to obtain an element of $\mathcal{U}(n+1,k)$. A further $k-1$ elements of $\mathcal{U}(n+1,k)$ may be obtained by appending $\{n+1\}$ to u and then inserting, in turn, $n+1$ into each of the $k-1$ already-existing parts of u. Thus far we have accounted for $kB(n,k-1)$ of the elements of $\mathcal{U}(n+1,k)$, bearing in mind that all the partitions generated in this way will be distinct.

Next, suppose that $u \in \mathcal{U}(n,k)$. We may obtain exactly $\binom{k}{2}$ elements of $\mathcal{U}(n+1,k)$ by inserting $n+1$ into one part of u and the other $n+1$ into a different part. It is also possible to put both of these identical elements into the same part of u. This leads, in total, to a further $\binom{k+1}{2} B(n,k)$ elements of $\mathcal{U}(n+1,k)$, noting that each of these is distinct from each of those generated in the previous paragraph.

Now let $\mathcal{V}(n,k-1)$ denote the set of all partitions of $\mathcal{M}(n,2)$ into exactly k nonempty (not necessarily distinct) parts. We consider some some $v \in \mathcal{V}(n,k-1)$ such that v has precisely one pair of identical parts. On appending the part $\{n+1\}$ to v and inserting the remaining $n+1$ into one of the identical parts, we will obtain an element of $\mathcal{U}(n+1,k)$. If the number of elements in each of the identical parts of v is j, then the remaining parts (excluding the singleton $\{n+1\}$) will comprise an element of a set isomorphic to $\mathcal{U}(n-j,k-3)$. Thus, in total, this process will give rise to

$$\sum_{j=1}^{n} \binom{n}{j} B(n-j,k-3)$$

elements of $\mathcal{U}(n+1,k)$, noting that there are $\binom{n}{j}$ possible choices of j elements from $\{1,2,3,\ldots,n\}$.

Now, if $v \in \mathcal{V}(n,k)$ is such that it has precisely one pair of identical parts, each with j elements, then the remaining parts form an element of a set isomorphic to $\mathcal{U}(n-j,k-2)$. If $n+1$ is inserted into one of the identical parts of v and, in turn, the remaining $n+1$ is placed into each of the other $k-2$ parts of v, then an element of $\mathcal{U}(n+1,k)$ will result. Another possibility here for generating an element of $\mathcal{V}(n+1,k)$ is to insert both of the elements $n+1$ into one of the identical parts of v. We then have a further

$$(k-1)\sum_{j=1}^{n} \binom{n}{j} B(n-j,k-2)$$

elements of $\mathcal{U}(n+1,k)$.

Finally, suppose that $v \in \mathcal{V}(n,k)$ is such that it has precisely two pairs of identical parts, one pair with l elements, the other with m (it is worth noting that it is impossible to form trios of identical parts, or beyond, using the elements from $\mathcal{M}(n,2)$). The remaining parts will then form an element of a set isomorphic to $\mathcal{U}(n-l-m, k-4)$. In such cases, the only way to obtain an element from $\mathcal{U}(n+1,k)$ would be to insert $n+1$ into one of the parts possessing an identical twin and the remaining $n+1$ into one of the parts comprising the other identical pair. This results in

$$\sum_{j=2}^{n} \binom{n}{j} S(j,2) B(n-j, k-4)$$

contributions to $\mathcal{U}(n+1,k)$, bearing in mind that the number of partitions of $l+m$ distinct elements from $\mathcal{M}(n,1) = \{1,2,3,\dots,n\}$ into two parts is given by $\binom{n}{l+m} S(l+m, 2)$.

Note that, although there certainly exists the possibility for an element $v \in \mathcal{V}(n,k)$ to contain more than two pairs of identical parts, it is not possible in such cases to insert the element $n+1$ and its twin into these parts in such a way as to create an element from $\mathcal{U}(n+1,k)$. ■

It is now a matter of obtaining formulas for $B(n,k)$ from the recurrence relation given in Theorem 1. To do this, we employ a particular family of exponential generating functions, defined as follows:

DEFINITION. The *exponential generating function* $G_k(x)$ for the sequence of column k of the number triangle $B(n,k)$ is defined by

$$G_k(x) = \sum_{n=0}^{\infty} \frac{B(n,k)x^n}{n!}.$$

We illustrate below how to calculate recursively the exponential generating function for the kth column of the number triangle $B(n,k)$, noting first that $G_1(x) = e^x - 1$.

THEOREM 2.

$$\begin{aligned} G_2(x) &= \frac{1}{2}(e^{3x} - 3e^x + 2) \\ &= \frac{1}{2}(e^x - 1)^2(e^x + 2), \end{aligned}$$

and hence, for $n \geq 1$,

$$B(n,2) = \frac{3}{2}(3^{n-1} - 1).$$

Proof. Theorem 1 gives, for $n \geq 1$, the recurrence relation

$$\begin{aligned} B(n+1,2) &= \binom{3}{2} B(n,2) + 2B(n,1) + B(0,0) \\ &= 3B(n,2) + 3. \end{aligned}$$

From this we obtain

$$\sum_{n=1}^{\infty} \frac{B(n+1,2)x^n}{n!} = 3\sum_{n=1}^{\infty} \frac{B(n,2)x^n}{n!} + 3\sum_{n=1}^{\infty} \frac{x^n}{n!},$$

leading to the following first-order linear differential equation:

$$G_2'(x) - 3G_2(x) = 3(e^x - 1),$$

which may be solved by using an integrating factor; see [**14**], for example. This gives, on noting that $G_2(0) = 0$,

$$G_2(x) = \frac{1}{2}(e^{3x} - 3e^x + 2),$$

as required. ■

Similarly, in order to obtain $G_3(x)$, we use Theorem 1 once more to give

$$\begin{aligned} B(n+1,3) &= \binom{4}{2} B(n,3) + 3B(n,2) + \sum_{j=1}^{n} \binom{n}{j} \{2B(n-j,1) + B(n-j,0)\} \\ &= 6B(n,3) + 3B(n,2) + 2\sum_{j=1}^{n-1} \binom{n}{j} + B(0,0) \\ &= 6B(n,3) + 3B(n,2) + 2^{n+1} - 3, \end{aligned}$$

where, in the second line above, we simply note that $B(n-n, j) = B(0,1) = 0$ and $B(n-j, 0) = 0$ unless $j = n$, in which case $B(n-j, 0) = 1$ by definition. Then

$$\sum_{n=1}^{\infty} \frac{B(n+1,3)x^n}{n!} = 6\sum_{n=1}^{\infty} \frac{B(n,3)x^n}{n!} + 3\sum_{n=1}^{\infty} \frac{B(n,2)x^n}{n!} + 2\sum_{n=1}^{\infty} \frac{(2x)^n}{n!} - 3\sum_{n=1}^{\infty} \frac{x^n}{n!},$$

leading to the differential equation

$$\begin{aligned} G_3'(x) - 6G_3(x) &= 3G_2(x) + 2(e^{2x} - 1) - 3(e^x - 1) \\ &= \frac{1}{2}(3e^{3x} + 4e^{2x} - 15e^x + 8), \end{aligned}$$

the solution of which is

$$\begin{aligned} G_3(x) &= \frac{1}{6}(e^{6x} - 3e^{3x} - 3e^{2x} + 9e^x - 4) \\ &= \frac{1}{6}(e^x - 1)^2(e^{4x} + 2e^{3x} + 3e^{2x} + e^x - 4). \end{aligned}$$

The recurrence relations for $B(n,k)$ clearly become somewhat more complicated for larger values of k. Nevertheless, it is possible to show that

$$\begin{aligned} G_4(x) &= \frac{1}{24}(e^{10x} - 4e^{6x} - 6e^{4x} + 12e^{3x} + 15e^{2x} - 28e^x + 10) \\ &= \frac{1}{24}(e^x - 1)^3(e^{7x} + 3e^{6x} + 6e^{5x} + 10e^{4x} + 11e^{3x} + 9e^{2x} - 2e^x - 10), \\ G_5(x) &= \frac{1}{120}(e^{15x} - 5e^{10x} - 10e^{7x} + 20e^{6x} + 30e^{4x} - 25e^{3x} - 75e^{2x} + 90e^x - 26) \\ &= \frac{1}{120}(e^x - 1)^4(e^{11x} + 4e^{10x} + 10e^{9x} + 20e^{8x} + 35e^{7x} + 51e^{6x} \\ &\qquad + 64e^{5x} + 70e^{4x} + 55e^{3x} + 25e^{2x} - 14e^x - 26), \end{aligned}$$

and so on, noting that $S(n,2) = 2^{n-1} - 1$; see [**12**]. From these generating functions we may obtain the following formulas, each valid for $n \geq 1$:

$$\begin{aligned}
B(n,3) &= \frac{1}{6}(6^n - 3^{n+1} - 3 \cdot 2^n + 9) \\
&= \frac{1}{2}(2 \cdot 6^{n-1} - 3^n - 2^n + 3), \\
B(n,4) &= \frac{1}{24}(10^n - 4 \cdot 6^n - 6 \cdot 4^n + 12 \cdot 3^n + 15 \cdot 2^n - 28) \\
&= \frac{1}{6}(25 \cdot 10^{n-2} - 6^n - 6 \cdot 4^{n-1} + 3^{n+1} + 15 \cdot 2^{n-2} - 7), \\
B(n,5) &= \frac{1}{120}(15^n - 5 \cdot 10^n - 10 \cdot 7^n + 20 \cdot 6^n + 30 \cdot 4^n - 25 \cdot 3^n - 75 \cdot 2^n + 90) \\
&= \frac{1}{24}(3 \cdot 15^{n-1} - 10^n - 2 \cdot 7^n + 4 \cdot 6^n + 6 \cdot 4^n - 5 \cdot 3^n - 15 \cdot 2^n + 18).
\end{aligned}$$

The last formula gives a rather large number for such a poorly-attended convention:

$$B(8,5) = 17005149.$$

Evaluating $A(n, k)$

This might be termed the "anything goes" enumeration, since both repeated parts and repeated elements within parts are allowed, and it is indeed clear that $A(n,k) \geq B(n,k)$ for all $n, k \in \mathbb{N}$. As we show here, rather than go through the whole rigmarole of the previous section, we can use the formulas for $B(n,k)$ to derive those for $A(n,k)$. In order to obtain a formula for $A(n,2)$, we note that

$$\mathcal{V}(n,2) = \mathcal{U}(n,2) \ \cup \ \{\{1,2,3,\ldots,n\},\{1,2,3,\ldots,n\}\}.$$

Thus

$$\begin{aligned}
A(n,2) &= B(n,2) + 1 \\
&= \frac{1}{2}(3^n - 1).
\end{aligned}$$

Next, let us consider $A(n,3)$. A typical element of the set $\mathcal{V}(n,3)\backslash\mathcal{U}(n,3)$ is of the form $\{a,a,b\}$, where $a \subset \{1,2,3,\ldots,n\}$ with $1 \leq |a| \leq n-1$, and b is the multiset such that $a \cup a \cup b = \mathcal{M}(n,2)$. The number of such elements is given by

$$\sum_{j=1}^{n-1} \binom{n}{j} = 2^n - 2.$$

Therefore,

$$\begin{aligned}
A(n,3) &= B(n,3) + 2^n - 2 \\
&= \frac{1}{6}(6^n - 3^{n+1} + 3 \cdot 2^n - 3) \\
&= \frac{1}{2}(2 \cdot 6^{n-1} - 3^n + 2^n - 1).
\end{aligned}$$

Continuing in this manner gives

$$\begin{aligned} A(n,4) &= B(n,4) + \sum_{j=1}^{n}\binom{n}{j}B(n-j,2) + S(n,2) \\ &= B(n,4) + \frac{1}{2}\sum_{j=1}^{n-1}\binom{n}{j}(3^{n-j}-3) + 2^{n-1} - 1 \\ &= B(n,4) + \frac{1}{2}(4^n - 3^n - 1) - \frac{3}{2}(2^n - 2) + 2^{n-1} - 1 \\ &= \frac{1}{24}(10^n - 4\cdot 6^n + 6\cdot 4^n - 9\cdot 2^n + 8) \\ &= \frac{1}{3}(125\cdot 10^{n-3} - 3\cdot 6^{n-1} + 3\cdot 4^{n-1} - 9\cdot 2^{n-3} + 1) \end{aligned}$$

and

$$\begin{aligned} A(n,5) &= B(n,5) + \sum_{j=1}^{n}\binom{n}{j}B(n-j,3) + \sum_{j=2}^{n}\binom{n}{j}S(j,2)B(n-j,1) \\ &= \frac{1}{120}(15^n - 5\cdot 10^n + 10\cdot 7^n - 30\cdot 4^n + 35\cdot 3^n - 15\cdot 2^n + 10) \\ &= \frac{1}{24}(3\cdot 15^{n-1} - 10^n + 2\cdot 7^n - 6\cdot 4^n + 7\cdot 3^n - 3\cdot 2^n + 2). \end{aligned}$$

In general, it is the case that

$$A(n,k) = B(n,k) + \sum_{i=1}^{\lfloor k/2\rfloor}\sum_{j=i}^{n}\binom{n}{j}S(j,i)B(n-j,k-2i),$$

easily allowing us to obtain a formula for the sequence of column k of the number triangle $A(n,k)$, once formulas have been derived for the sequences of the first k columns of the number triangle $B(n,k)$. We have

$$A(8,5) = 17656653.$$

It is worth noting that, in relative terms at least, this is barely more than $B(8,5)$.

Calculating $D(n,k)$ and $C(n,k)$

A recurrence relation for $D(n,k)$ may be derived in a similar manner to that used to calculate the one for $B(n,k)$ in Theorem 1. This result is stated below without proof; indeed, the interested reader might like to derive it for themselves.

THEOREM 3. *For $n \geq 0$,*

$$D(n+1,k) = \binom{k}{2}D(n,k) + (k-1)D(n,k-1) + r(n,k) + s(n,k),$$

where

$$r(n,k)=\sum_{j=1}^{n}\binom{n}{j}\{(k-2)D(n-j,k-2)+D(n-j,k-3)\},$$

$$s(n,k)=\sum_{j=2}^{n}\binom{n}{j}S(j,2)D(n-j,k-4)$$

and $D(0,0)=1$ *by definition.*

It is then possible to obtain exponential generating functions for the columns of the number triangle $D(n,k)$. Just as we were able to obtain the number triangle $A(n,k)$ from that of $B(n,k)$, we may, using

$$C(n,k)=D(n,k)+\sum_{i=1}^{\lfloor k/2\rfloor}\sum_{j=i}^{n}\binom{n}{j}S(j,i)D(n-j,k-2i),$$

derive the number triangle $C(n,k)$ from that of $D(n,k)$. Comparing the results

$$C(8,5)=768747 \qquad \text{and} \qquad D(n,k)=759045$$

with those for $A(8,5)$ and $B(8,5)$ we see, in this particular example at least, that insisting that no pair of identical twins sits at the same table is far more restrictive than insisting that the groups sitting at any pair of tables are distinguishable from one another.

Further results

In this final section, we gather together further results in connection with our twin-convention problem. These are concerned with ordinary generating functions, asymptotic relations, and ordered partitions.

DEFINITION. The *ordinary generating function* $\overline{G}_k(x)$ for the sequence of column k of the number triangle $B(n,k)$ is defined by

$$\overline{G}_k(x)=\sum_{n=0}^{\infty}B(n,k)x^n.$$

We may obtain $\overline{G}_k(x)$ easily from the previously-obtained formulas for $B(n,k)$. For example, since

$$B(n,3)=\frac{1}{6}(6^n-3^{n+1}-3\cdot 2^n+9),$$

it is the case that

$$\begin{aligned}\overline{G}_3(x)&=\frac{1}{6}\left(\frac{6x}{1-6x}-\frac{9x}{1-3x}-\frac{6x}{1-2x}+\frac{9x}{1-x}\right)\\&=\frac{x^2(-24x^2+8x+1)}{(1-x)(1-2x)(1-3x)(1-6x)}.\end{aligned}$$

Next, it can be shown that the asymptotic relations

$$B(n,k) \sim \frac{[k(k+1)]^n}{2^n k!} \qquad \text{and} \qquad D(n,k) \sim \frac{[k(k-1)]^n}{2^n k!}$$

hold for fixed $k \geq 1$ and $k \geq 3$, respectively, and hence that

$$\frac{B(n,k)}{D(n,k)} \sim \left(\frac{k+1}{k-1}\right)^n$$

for fixed $k \geq 3$. It is worth noting, for fixed k, that

$$\frac{B(n,k)}{D(n,k)} \to \infty \qquad \text{as} \qquad n \to \infty.$$

Those interested might like to come up with a heuristic explanation of this result.

It is also possible to obtain an expression for the number of ordered partitions $F(n,2)$ of $\mathcal{M}(n,2)$, where there are no restrictions regarding repeated parts or repeated elements within parts. This corresponds to the situation in which the tables are distinguishable. First, note that the number of multi-valued functions from $\mathcal{M}(n,2)$ to $\{1,2,3,\ldots,k\}$ is given by $\binom{k+1}{2}^n$, and that this may also be evaluated as

$$\sum_{m=1}^{k} \binom{k}{m} E(n,m),$$

where $E(n,m)$ is the number of surjective mappings from $\mathcal{M}(n,2)$ to $\{1,2,3,\ldots,m\}$. Binomial inversion then gives

$$E(n,m) = \sum_{i=1}^{m} \binom{m}{i} \binom{i+1}{2}^n (-1)^{m-i}.$$

Finally, summing from $m=1$ to $m=2n$ in order to cover all possible partition sizes, it follows that

$$F(n,2) = \sum_{m=1}^{2n} \sum_{i=1}^{m} \binom{m}{i} \binom{i+1}{2}^n (-1)^{m-i}.$$

I have performed similar calculations for triplet conventions. (Yes, I promise that they do exist; I checked!) This involved enumerating the partitions of $\mathcal{M}(n,3)$ into exactly k nonempty parts. However, the recurrence relations are much more complicated than for the twin-convention scenario because more cases need to be accounted for. The ideas presented in this paper can, in fact, be extended to deal with the enumeration of partitions of $\mathcal{M}(n,j)$ into exactly k nonempty parts for any $j \in \mathbb{N}$—in theory at least!

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Summary In this article we consider a particular combinatorial scenario involving n sets of identical twins. We show how, subject to various assumptions and conditions on the possible groupings, formulas may be obtained in terms of n for the number of ways in which these $2n$ individuals can be seated at k tables for any fixed value of k. This is achieved by deriving recurrence relations and subsequently employing exponential generating functions.

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The Lah Numbers and the nth Derivative of $e^{1/x}$

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What do you get when you take the derivative of $e^{1/x}$ n times? The result must be of the form $e^{1/x}$ times a sum of various powers of x^{-1}, but what, exactly, are the coefficients of those powers?

In this article we show that these coefficients are the *Lah numbers*, a triangle of integers whose best-known applications are in combinatorics and finite difference calculus. We give five proofs, using four different properties of the Lah numbers. In the process we take a tour through several areas of mathematics, seeing the binomial coefficients, Faà di Bruno's formula, set partitions, Maclaurin series, factorial powers, the Poisson probability distribution, and hypergeometric functions.

The first proof: the direct approach

First we step through the process whereby we might construct a conjecture for the nth derivative of $e^{1/x}$. The natural thing to do is to calculate the derivative for several small values of n and look for a pattern. This leads to the following.

n	$D^n(e^{1/x})$
1	$-e^{1/x}\ x^{-2}$
2	$e^{1/x}(2x^{-3} + x^{-4})$
3	$-e^{1/x}(6x^{-4} + 6x^{-5} + x^{-6})$
4	$e^{1/x}(24x^{-5} + 36x^{-6} + 12x^{-7} + x^{-8})$
5	$-e^{1/x}(120x^{-6} + 240x^{-7} + 120x^{-8} + 20x^{-9} + x^{-10})$

The most famous triangle of numbers, of course, is the triangle of binomial coefficients $\binom{n}{k}$, which starts like so:

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n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

Ignoring signs, the coefficients in the table of derivatives of $e^{1/x}$ look suspiciously like multiples of the binomial coefficients. Following up with this conjecture, and letting $c_{n,k}$ be the coefficient of x^{-n-k} in the nth derivative of $e^{1/x}$, we get this table.

n	$c_{n,0}/\binom{n}{0}$	$c_{n,1}/\binom{n}{1}$	$c_{n,2}/\binom{n}{2}$	$c_{n,3}/\binom{n}{3}$	$c_{n,4}/\binom{n}{4}$	$c_{n,5}/\binom{n}{5}$
0	1					
1	0	1				
2	0	1	1			
3	0	2	2	1		
4	0	6	6	3	1	
5	0	24	24	12	4	1

These numbers also look like they might be multiples of the binomial coefficients, just shifted in n and k. Conjecturing that these numbers are multiples of $\binom{n-1}{k-1}$, we have the following.

n	$\frac{c_{n,0}}{\binom{n}{0}}$	$\frac{c_{n,1}}{\binom{n}{1}\binom{n-1}{0}}$	$\frac{c_{n,2}}{\binom{n}{2}\binom{n-1}{1}}$	$\frac{c_{n,3}}{\binom{n}{3}\binom{n-1}{2}}$	$\frac{c_{n,4}}{\binom{n}{4}\binom{n-1}{3}}$	$\frac{c_{n,5}}{\binom{n}{5}\binom{n-1}{4}}$
0	1					
1	0	1				
2	0	1	1			
3	0	2	1	1		
4	0	6	2	1	1	
5	0	24	6	2	1	1

This triangle of numbers we immediately recognize to be the factorials, with (except for column $k = 0$) the number in entry n, k as $(n - k)!$. This pattern seems too regular to be a coincidence, and in fact, the pattern continues.

THEOREM 1.

$$\frac{d^n}{dx^n}(e^{1/x}) = (-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!\, x^{-n-k}.$$

Proof. This looks like a prime candidate for induction. We have already shown that the expression holds for $n = 1, 2, 3, 4$, and 5. Then

$$\frac{d^{n+1}}{dx^{n+1}}(e^{1/x}) = \frac{d}{dx}\left((-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!\, x^{-n-k}\right)$$

$$= (-1)^{n+1} e^{1/x} \left(\sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!\, x^{-n-k-2}\right.$$

$$\left. + \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)!\,(n+k)x^{-n-k-1}\right).$$

Shifting indices on the first sum and using the fact that $\binom{n}{-1} = \binom{n}{n+1} = 0$, we can combine the two sums into one:

$$\sum_{k=1}^{n+1} \left(\binom{n}{k-1}\binom{n-1}{k-2}(n-k+1)! + \binom{n}{k}\binom{n-1}{k-1}(n-k)!\,(n+k) \right) x^{-n-k-1}.$$

Focusing now on the binomial expression in the summand, we have

$$\begin{aligned}
&\binom{n}{k-1}\binom{n-1}{k-2}(n-k+1)! + \binom{n}{k}\binom{n-1}{k-1}(n-k)!\,(n+k) \\
&\quad = \frac{n!\,(n-1)!}{(k-1)!\,(k-2)!\,(n+1-k)!} + \frac{n!\,(n-1)!\,(n+k)}{k!\,(k-1)!\,(n-k)!} \\
&\quad = \frac{(n+1)!\,n!}{k!\,(k-1)!\,(n+1-k)!}\left(\frac{k(k-1)}{n(n+1)} + \frac{(n+k)(n+1-k)}{n(n+1)}\right) \\
&\quad = \binom{n+1}{k}\binom{n}{k-1}(n+1-k)!\left(\frac{k^2-k+n^2-k^2+n+k}{n(n+1)}\right) \\
&\quad = \binom{n+1}{k}\binom{n}{k-1}(n+1-k)!,
\end{aligned}$$

completing the proof. ∎

The Lah numbers

These numbers, $L(n,k) = \binom{n}{k}\binom{n-1}{k-1}(n-k)!$, have been studied in other contexts. The Slovenian mathematician Ivo Lah first investigated their properties in a pair of papers [**7, 8**] from the 1950s, and as a result they are called the *Lah numbers.* They appear as sequence A008297 in the On-line Encyclopedia of Integer Sequences [**10**]. The following table contains the first several rows of the Lah number triangle.

	Lah numbers					
$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	1				
3	6	6	1			
4	24	36	12	1		
5	120	240	120	20	1	
6	720	1800	1200	300	30	1

Besides their representation $\binom{n}{k}\binom{n-1}{k-1}(n-k)!$ in terms of binomial coefficients, some of the better-known properties of the Lah numbers $L(n,k)$ are given in the next theorem.

THEOREM 2.

1. *The Lah numbers satisfy a nice triangular recurrence relation:* $L(n+1,k) = (n+k)L(n,k) + L(n,k-1)$, *with* $L(n,0) = 0$, $L(n,k) = 0$ *for* $n < k$, *and* $L(1,1) = 1$.
2. *The Lah numbers are the coefficients used to convert rising factorial powers to falling factorial powers:* $x^{\overline{n}} = \sum_{k=1}^{n} L(n,k)x^{\underline{k}}$, *where* $x^{\overline{n}} = x(x+1)\cdots$

$(x+n-1)$ and $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$. (This was the property that Lah was originally studying.)

3. *The Lah number $L(n,k)$ counts the number of ways a set of n elements can be partitioned into k nonempty tuples.*
4. *The Lah numbers can be represented in terms of Stirling numbers: $L(n,k) = \sum_{j=k}^{n} {n \brack j}{j \brace k}$, where ${n \brack j}$ and ${j \brace k}$ are Stirling numbers of the first and second kinds, respectively.*

We use the first three of these properties in subsequent proofs that the Lah numbers are the coefficients in the nth derivative of $e^{1/x}$.

Proof. We take $L(n,k) = \binom{n}{k}\binom{n-1}{k-1}(n-k)!$ as the definition of the Lah numbers.

Property 3. To construct k nonempty tuples, first choose a permutation of n elements. This can be done in $n!$ ways. Then choose $k-1$ of the $n-1$ possible cut points in the permutation to create k nonempty tuples. This can be done in $\binom{n-1}{k-1}$ ways. However, since there are $k!$ ways to order the tuples, there are $k!$ permutations that create the same set of k nonempty tuples. Thus the number of ways a set of n elements can be partitioned into k nonempty tuples is $\frac{n!}{k!}\binom{n-1}{k-1} = \binom{n}{k}\binom{n-1}{k-1}(n-k)! = L(n,k)$.

Property 1 from Property 3. One way to count the number of ways of constructing k tuples from $n+1$ elements is to condition on element $n+1$. The number of ways to construct k tuples from $n+1$ elements in which element $n+1$ is in a tuple by itself is $L(n,k-1)$. Otherwise, element $n+1$ could be placed in front of any of n elements in k already-existing tuples, in n ways, or at the end of one of the tuples, in k ways. There are $(n+k)L(n,k)$ ways to do this. Clearly, it is impossible to construct 0 nonempty tuples from n elements, it is impossible to construct more tuples than elements, and there is one way to construct a nonempty tuple from one element.

Property 2 from Property 3. First, assume that x is a positive integer. We show that the two sides of the equation in Property 2 count the number of ways to construct x (not necessarily nonempty) ordered tuples from n elements. For the left side, construct the tuples by placing elements one-by-one in the x possible tuples. If there are j elements already placed, then element $j+1$ can be placed in front of any of the already-placed elements, in j ways, or it can be placed at the end of one of the x tuples, in x ways, for a total of $x+j$ possible placements. Thus the number of ways to construct the x tuples is $x(x+1)\cdots(x+n-1) = x^{\overline{n}}$. For the right side, condition on the number of nonempty tuples. There are $L(n,k)$ ways to form the k nonempty tuples from the n elements, and then there are $x(x-1)\cdots(x-k+1) = x^{\underline{k}}$ ways to order the tuples (including the empty ones). Summing up over the possible values of k yields $\sum_{k=1}^{n} L(n,k)x^{\underline{k}}$, which then must also be the total number of ways to construct x ordered tuples from n elements. This proves Property 2 for x a positive integer. However, the equation in Property 2 is a polynomial of degree n. Since it takes $n+1$ values to specify such a polynomial, and we have just shown that this equation holds for an infinite number of values of x, Property 2 must hold for all real values of x.

Property 4 from Property 2. Stirling numbers of the first kind ${n \brack k}$ are the coefficients when converting from rising powers to ordinary powers, and Stirling numbers of the second kind ${n \brace k}$ are the coefficients when converting from ordinary powers to falling powers [**5**, p. 264]. Thus we have $x^{\overline{n}} = \sum_{j=1}^{n} {n \brack j} x^j = \sum_{j=1}^{n} {n \brack j} \sum_{k=1}^{j} {j \brace k} x^{\underline{k}} = \sum_{k=1}^{n}\sum_{j=k}^{n} {n \brack j}{j \brace k} x^{\underline{k}}$, and so $L(n,k) = \sum_{j=k}^{n} {n \brack j}{j \brace k}$. ■

(See also Petkovšek and Pisanski [**11**] for proofs of Properties 1, 2, and 3.)

We now present four more proofs that

$$\frac{d^n}{dx^n}(e^{1/x}) = (-1)^n e^{-1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}.$$

The first three use Properties 1, 2, and 3, respectively, of the Lah numbers. The last proof uses Kummer's confluent hypergeometric function transformation, as well as the representation of the Lah numbers in terms of binomial coefficients in Theorem 1.

The second proof: use the recurrence relation

The recurrence relation given in Property 1 should greatly simplify our first induction proof. For the induction step, we have

$$\begin{aligned} \frac{d^{n+1}}{dx^{n+1}}(e^{1/x}) &= \frac{d}{dx}\left((-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}\right) \\ &= (-1)^{n+1} e^{1/x} \left(\sum_{k=1}^{n} L(n,k) x^{-n-k-2} + \sum_{k=1}^{n} L(n,k)(n+k) x^{-n-k-1}\right) \\ &= (-1)^{n+1} e^{1/x} \sum_{k=1}^{n+1} \left(L(n,k-1) + (n+k)L(n,k)\right) x^{-n-k-1} \\ &= (-1)^{n+1} e^{1/x} \sum_{k=1}^{n+1} L(n+1,k) x^{-(n+1)-k}. \end{aligned}$$

(In fact, we can now see that our first proof was essentially a proof of Property 1 of the Lah numbers.)

The third proof: factorial moments of the Poisson distribution

Our third proof uses the Poisson probability distribution and the Maclaurin series for e^x. The Poisson distribution is often used to model the number of occurrences of some event over a fixed period of time, such as the number of arrivals to a fast-food restaurant in an hour or the number of salmon that swim by a particular point on a river in a day. The probability mass function for a Poisson distribution is $f(j) = \lambda^j e^{-\lambda}/j!$, for $j = 0, 1, \ldots$. Here λ is a rate parameter; for example, λ might represent the expected number of occurrences in the given time interval.

To begin our third proof, consider the Maclaurin series for e^x:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Substituting x^{-1} for x and differentiating n times yields

$$\frac{d^n}{dx^n}(e^{1/x}) = (-1)^n x^{-n} \sum_{j=0}^{\infty} \frac{j^{\overline{n}} x^{-j}}{j!}.$$

The summand is similar to the probability mass function of a Poisson distribution with $\lambda = x^{-1}$ for the rate parameter. In fact, with an $e^{-1/x}$ in the summand, the infinite series is exactly the nth rising factorial moment $E[X^{\overline{n}}]$. While the rising factorial moments of a Poisson distribution do not have a known simple expression, it is known that the falling factorial moments do, and this is easily proved. If Y is Poisson, then

$$E[Y^{\underline{n}}] = \sum_{j=0}^{\infty} \frac{j^{\underline{n}}\lambda^j e^{-\lambda}}{j!} = e^{-\lambda}\sum_{j=n}^{\infty} \frac{j^{\underline{n}}\lambda^j}{j!} = e^{-\lambda}\sum_{j=n}^{\infty} \frac{j!\lambda^j}{(j-n)!j!}$$
$$= e^{-\lambda}\sum_{i=0}^{\infty} \frac{\lambda^{i+n}}{i!} = e^{-\lambda}\lambda^n e^{\lambda} = \lambda^n.$$

Given this, it makes sense to convert rising powers to falling powers. Doing so, while using Property 2 of the Lah numbers to make the conversion, yields

$$\frac{d^n}{dx^n}(e^{1/x}) = (-1)^n x^{-n} \sum_{j=0}^{\infty}\sum_{k=1}^{n} L(n,k)\frac{j^{\underline{k}}x^{-j}}{j!}$$
$$= (-1)^n x^{-n} e^{1/x} \sum_{k=1}^{n} L(n,k)E[X^{\underline{k}}]$$
$$= (-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k)x^{-n-k},$$

where X is Poisson (with rate parameter $\lambda = x^{-1}$) and $E[X^{\underline{k}}] = x^{-k}$.

The fourth proof: Faà di Bruno's formula

Our fourth proof uses a formula named after Faà di Bruno for the nth derivative of a composite function, as well as the combinatorial interpretation of the Lah numbers (Property 3). Faà di Bruno's papers [**2**] and [**3**] mentioning his formula date from the mid-1800s. However, neither contains a proof of the formula (merely its statement), and Faà di Bruno was not actually the first to state the formula or to prove it! Readers interested in learning more should consult Warren Johnson's paper "The curious history of Faà di Bruno's formula" [**6**].

Faà di Bruno's formula can be represented in multiple forms [**6**]; for instance, there is a combinatorial version, a version that uses Bell polynomials, and a determinantal version. The most useful for our purposes is the combinatorial version:

$$\frac{d^n}{dx^n} f(g(x)) = \sum f^{(k)}(g(x))(g'(x))^{s_1}(g''(x))^{s_2}\cdots(g^{(n-k+1)}(x))^{s_{n-k+1}}. \tag{1}$$

Here the sum is over all partitions of $\{1, 2, \ldots, n\}$ into nonempty sets, where k is the number of sets in the partition and s_i is the number of sets with exactly i elements.

Why, exactly, does the complicated expression on the right of Equation (1) give the nth derivative of the composite function $f(g(x))$? An induction proof can be given using the following ideas. Each partition of $\{1, 2, \ldots, n+1\}$ can be formed in exactly one way by adding $n+1$ to a partition of $\{1, 2, \ldots, n\}$. If we add $\{n+1\}$ as a singleton set, that increases the total number of sets by 1 and the number of sets containing one element by 1. This corresponds to differentiating $f^{(k)}(g(x))$ to get $f^{(k+1)}(g(x))g'(x)$. If we add $n+1$ to an already-existing set containing i elements, that decreases the number of sets containing i elements by 1 and increases the number of sets containing $i+1$ elements by 1. With s_i sets containing i elements, this corresponds to differentiating $(g^{(i)}(x))^{s_i}$ to get $(g^{(i)}(x))^{s_i-1}g^{(i+1)}(x)$. Thus differentiating the right-hand side of Equation (1) corresponds to all the different ways to obtain a partition of $\{1, 2, \ldots, n+1\}$ by adding $n+1$ to a partition of $\{1, 2, \ldots, n\}$.

Since $f(x) = e^x$, and $g(x) = 1/x$, Faà di Bruno's formula tells us that

$$\frac{d^n}{dx^n}(e^{1/x}) = \sum e^{1/x}(-x^{-2})^{s_1}(2x^{-3})^{s_2}\cdots((-1)^{n-k+1}(n-k+1)!\,x^{-n+k-2})^{s_{n-k+1}}.$$

This expression looks fairly complicated, but it simplifies nicely. Factor out $e^{1/x}$ and then regroup the rest by powers of -1, powers of x, and factorials, so that the summand is the product of the following three factors:

1. $(-1)^{s_1+2s_2+\cdots+(n-k+1)s_{n-k+1}}$,
2. $x^{-(2s_1+3s_2+\cdots+(n-k+2)s_{n-k+1})}$, and
3. $(1!)^{s_1}(2!)^{s_2}\cdots((n-k+1)!)^{s_{n-k+1}}$.

For a given partition P_k of n elements into k nonempty sets, $s_1 + 2s_2 + \cdots + (n-k+1)s_{n-k+1}$ is the total number of elements in P_k, and so the first factor simplifies to $(-1)^n$. Similarly, $s_1 + s_2 + \cdots + s_{n-k+1}$ is the total number of sets in P_k, and so the second factor simplifies to x^{-n-k}. Finally, each set of size i in P_k can have its elements permuted in $i!$ ways to create a tuple, and so the third factor is the number of partitions of $\{1, 2, \ldots, n\}$ into nonempty tuples that can be formed from the partition P_k of $\{1, 2, \ldots, n\}$ into nonempty sets. Therefore,

$$\begin{aligned}\frac{d^n}{dx^n}(e^{1/x}) &= (-1)^n e^{1/x}\sum_{k=1}^{n} x^{-n-k}\sum_{P_k}(1!)^{s_1}(2!)^{s_2}\cdots((n-k+1)!)^{s_{n-k+1}}\\ &= (-1)^n e^{1/x}\sum_{k=1}^{n} L(n,k)x^{-n-k}.\end{aligned}$$

The fifth proof: Kummer's hypergeometric transformation

Our last proof uses properties of hypergeometric functions—especially Kummer's hypergeometric transformation—in addition to the binomial coefficient expression for the Lah numbers.

A general *hypergeometric function* is of the form

$$\sum_{k=0}^{\infty}\frac{a_1^{\overline{k}}\cdots a_m^{\overline{k}}}{b_1^{\overline{k}}\cdots b_n^{\overline{k}}}\frac{z^k}{k!}.$$

As such, it is a power series in z with the a_i's and b_i's as parameters. A perhaps surprisingly large number of functions can be expressed as hypergeometric series, often through their Taylor expansions. The study of hypergeometric series goes back at least to Euler, and Gauss proved some of their properties in his doctoral dissertation. Graham, Knuth, and Patashnik's *Concrete Mathematics* [**5**, Ch. 5] contains a nice introduction to hypergeometric functions.

The version of the hypergeometric series we need is the one in which m and n are both 1, i.e.,

$$\sum_{k=0}^{\infty}\frac{a^{\overline{k}}}{b^{\overline{k}}}\frac{z^k}{k!}.$$

This series is known as *Kummer's confluent hypergeometric function* and is denoted ${}_1F_1(a; b; z)$ [**4**, §13.1], $F\left({a \atop b} \mid z\right)$, or $M(a, b, z)$ [**5**, p. 206]. One of the most important properties of Kummer's confluent hypergeometric function is the transformation [**5**, Exercise 5.29]

$$M(a, b, z) = e^z M(b - a, b, -z). \tag{2}$$

For our final proof, as in the third proof, we start with the Maclaurin series for e^x. Substituting x^{-1} for x and differentiating n times yields

$$\frac{d^n}{dx^n}(e^{1/x}) = (-1)^n x^{-n} \sum_{k=0}^{\infty} \frac{k^{\overline{n}} x^{-k}}{k!}.$$

The sum can be expressed as a hypergeometric series:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{k^{\overline{n}} x^{-k}}{k!} &= \sum_{k=1}^{\infty} \frac{(n+k-1)!\, x^{-k}}{k!\,(k-1)!} \\ &= x^{-1} \sum_{k=0}^{\infty} \frac{(n+k)!\, x^{-k}}{(k+1)!\, k!} = x^{-1} n! \sum_{k=0}^{\infty} \frac{(n+1)^{\overline{k}}\, x^{-k}}{2^{\overline{k}}\, k!}.\end{aligned}$$

Thus the series is of the form of Kummer's confluent hypergeometric function $M(n+1, 2; 1/x)$. Applying Kummer's transformation (2), we have

$$\begin{aligned}\frac{d^n}{dx^n}(e^{1/x}) &= (-1)^n x^{-n-1} n! M(n+1, 2, 1/x) \\ &= (-1)^n x^{-n-1} n!\, e^{1/x} M(1-n, 2, -1/x) \\ &= (-1)^n x^{-n-1} n!\, e^{1/x} \sum_{k=0}^{\infty} \frac{(1-n)^{\overline{k}}\, (-x)^{-k}}{2^{\overline{k}}\, k!}.\end{aligned}$$

Since $(1-n)^{\overline{k}} = 0$ for $k > n-1$, we obtain

$$\begin{aligned}\frac{d^n}{dx^n}\left(e^{1/x}\right) &= (-1)^n x^{-n-1} n! e^{1/x} \sum_{k=0}^{n-1} \frac{(-1)^k (n-1)^{\underline{k}}\, (-1)^k x^{-k}}{(k+1)!\, k!} \\ &= (-1)^n x^{-n-1} e^{1/x} \sum_{k=0}^{n-1} \frac{n!\,(n-1)!\, x^{-k}}{(n-1-k)!\,(k+1)!\, k!} \\ &= (-1)^n x^{-n} e^{1/x} \sum_{k=1}^{n} \frac{n!\,(n-1)!\, x^{-k}}{(n-k)!\, k!\,(k-1)!} \\ &= (-1)^n e^{1/x} \sum_{k=1}^{n} \binom{n}{k}\binom{n-1}{k-1}(n-k)! x^{-n-k} \\ &= (-1)^n e^{1/x} \sum_{k=1}^{n} L(n,k) x^{-n-k}.\end{aligned}$$

For a unified treatment on evaluating hypergeometric sums, see Petkovšek, Wilf, and Zeilberger's text $A = B$ [**12**].

Note. After this paper was accepted for publication the authors found Theorem 1 in Comtet's *Advanced Combinatorics* [**1**, Ch. III, Ex. 7, p. 158]. Comtet mentions more general results as well, including the cases $e^{\sqrt{x}}$ and e^{x^2}.

Acknowledgment This paper is based on a discussion [**9**] between the four authors on the mathematics question-and-answer web site, Mathematics Stack Exchange. The authors have never met each other.

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Summary We give five proofs that the coefficients in the nth derivative of $\exp(1/x)$ are the Lah numbers, a triangle of integers whose best-known applications are in combinatorics and finite difference calculus. Our proofs use tools from several areas of mathematics, including binomial coefficients, Faà di Bruno's formula, set partitions, Maclaurin series, factorial powers, the Poisson probability distribution, and hypergeometric functions.

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NOTES

The Equation $f(2x) = 2f(x)f'(x)$

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The problem

While reading a paper on cosmology, a colleague found the claim (made without justification) that x and $\sinh x$ are the only real solutions of the nonlinear differential equation

$$f(2x) = 2f(x)f'(x), \quad f'(0) = 1 \tag{1}$$

on $[0, +\infty)$ that are positive when x is positive (in the paper, x is the distance from the origin, hence our restriction that $x \geq 0$). Before we investigate this claim, we remark that as this equation is not of the form $f'(x) = F(x, f(x))$, it is not (in the usual terminology) a differential equation. It can be converted into a delay differential equation, which is of the form $f'(x) = F\big(x, f(x), f(x+a)\big)$ for some constant a, but we have been unable to use the theory of such equations to good effect. Nevertheless, for brevity we shall refer to it as a differential equation.

We shall call a solution f of (1) a *positive solution* if $f(x) > 0$ when $x > 0$. Now a moment's thought shows that the question raised above is not the right question, because if f is a positive solution, then so is $c^{-1}f(cx)$ for every positive c (we need c to be positive because $f(x)$ is only defined for $x \geq 0$). Since $\sinh x$ is a positive solution, so is $c^{-1}\sinh(cx)$ when $c > 0$, and if we let $c \to 0$ we obtain the positive solution x. This suggests that we let

$$S(x, c) = \begin{cases} c^{-1}\sinh(cx) & \text{if } c > 0, \\ x & \text{if } c = 0, \end{cases}$$

and then rephrase the original question as follows: are the functions $S(x, c)$, $c \geq 0$, the only positive solutions of (1)? This question is still open, and here we give some partial results. We also refer to reader to [**2**], where formal power series solutions of the more general equation $f(kz) = kf(z)f'(z)$, where z and k are complex numbers, are discussed.

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Formal power series solutions

It is natural to begin by discussing formal power series solutions to (1) about the point 0.

THEOREM 1. *The only formal power series solutions of* (1) *are* x, $c^{-1}\sinh(cx)$, *and* $c^{-1}\sin(cx)$, *where* $c \neq 0$. *Of these, only the functions* $S(x, c)$ *are positive solutions of* (1).

Proof. Let f be a formal power series solution of (1), and write $f(x) = 0 + a_1 x + a_2 x^2 + \cdots$ (even though we know that $a_1 = 1$). Since

$$f(2x) = \frac{d}{dx}\big[f(x)\big]^2,$$

we see that

$$\sum_{n=1}^{\infty} a_n \big(2x\big)^n = \sum_{n=1}^{\infty}(n+1)[a_1 a_n + \cdots + a_n a_1]x^n.$$

If we now equate the coefficients of x^2 and use $a_1 = 1$, we obtain $a_2 = 0$. Now for $n \geq 3$, the expression $a_1 a_n + \cdots + a_n a_1$ contains terms other than $a_1 a_n$ and $a_n a_1$ so, for $n \geq 3$, we have

$$\big[2^n - 2(n+1)\big]a_n = (n+1)[a_2 a_{n-1} + \cdots + a_{n-1} a_2].$$

Now for $n = 3$, the coefficient of a_3 on the left is zero, so a_3 is not yet determined. However, if $n \geq 4$, then

$$2^n = (1+1)^n = 1 + n + \cdots + n + 1 > 2(n+1),$$

and we see that for each $n \geq 4$, the coefficient a_n is determined, uniquely and inductively, by the values $a_2, a_3, \ldots, a_{n-1}$. In conclusion, *for each real number b, there is a unique formal power series solution* $f(x) = x + bx^3 + \cdots$ *of* (1). Clearly these solutions are, for different b,

$$f(x) = \begin{cases} x & \text{if } b = 0, \\ c^{-1}\sinh(cx) & \text{if } b > 0 \text{ and } c = \sqrt{6b}, \\ c^{-1}\sin(cx) & \text{if } b < 0 \text{ and } c = \sqrt{-6b}, \end{cases}$$

and the proof is complete. ■

The differentiability of solutions

Next, we prove the following result.

THEOREM 2. *Any positive solution* f *of* (1) *is infinitely differentiable on* $(0, +\infty)$.

Proof. Let $E = (0, +\infty)$. Since f is differentiable on E, and $f(x) > 0$ there, we see that $f(2x)/2f(x)$ is differentiable on E. Thus $f'(x)$ is differentiable on E. If we now differentiate both sides of (1), we obtain $f'(2x) = f(x)f''(x) + [f'(x)]^2$, and this shows (again because $f > 0$) that f'' is differentiable on E. More generally, if we continue in this way (using Leibnitz' theorem for the kth derivative of a product), we see that all derivatives of f exist on E; that is, f is infinitely differentiable on E. ■

Theorems 1 and 2 indicate, perhaps, why the general question is difficult to answer; for it shows that if there is a positive solution of (1) that is not $S(x, c)$ for any c, then it is infinitely differentiable on $(0, +\infty)$, but not a power series about $x = 0$. Of course, the derivative $f'(0) = 1$ in (1) is to be interpreted as a one-sided derivative, and it is not clear whether the higher-order one-sided derivatives of a solution f need exist at 0.

The lower envelope of solutions

The power series expansion of $\sinh x$ shows that if $x \geq 0$ and $0 \leq c \leq d$, then $x = S(x, 0) \leq S(x, c) \leq S(x, d)$; that is, the positive solutions $S(x, c)$ are ordered in the same way that c is ordered in $[0, +\infty)$. In particular, the solution x is the lower envelope of the solutions $S(x, c)$ for $c \geq 0$. We shall now show that *the solution x is the lower envelope of all positive solutions of* (1).

THEOREM 3. *The function x is the smallest positive solution of* (1).

Proof. Let f be a positive solution of (1). The key idea is to convert the differential equation into an integral equation, and then use the integral equation to prove the result. First, since $f(x)$ is differentiable for $x \geq 0$, with $f(0) = 0$, we have

$$f(x)^2 = \int_0^x \frac{d}{dt}\left[f(t)^2\right] dt = \int_0^x f(2t)\, dt = \frac{1}{2}\int_0^{2x} f(s)\, ds;$$

thus, any positive solution f of (1) satisfies the integral equation

$$f(x)^2 = \frac{1}{2}\int_0^{2x} f(t)\, dt. \tag{2}$$

For the moment, suppose that $f(x) \geq Ax$ for all positive x. Then, by replacing $f(t)$ by At in (2) and integrating, we see that $f(x) \geq \sqrt{A}x$. By repeating this argument, and noting that $A^{1/n} \to 1$ as $n \to +\infty$, we see that $f(x) \geq x$, so that Theorem 3 is proved once we can show that *for some A*, $f(x) \geq Ax$ for all positive x. This, however, is easy. By (1) or (2), f is strictly increasing on $(0, \infty)$. Thus $f(2x) > f(x)$, and hence, from (1), $f'(x) \geq 1/2$ there. Since $f(0) = 0$, this implies that $f(x) \geq x/2$ for all positive x. ∎

Exponential growth

Our last result shows that, apart from the solution x, any positive solution of (1) must exhibit some kind of exponential growth. More precisely, we prove the following result.

THEOREM 4. *Suppose that f is a positive solution of* (1), *but not the function x. Then there is a positive c, and a sequence x_n tending to $+\infty$, such that $f(x_n)/S(x_n, c) \to +\infty$.*

Proof. Let f be a positive solution of (1), and for $c \geq 0$ let

$$M(c) = \sup_{x \geq 0} \frac{f(x)}{S(x, c)} \leq +\infty.$$

Since $c \leq d$ implies $S(x, c) \leq S(x, d)$ for all x, we see that the map $c \mapsto M(c)$ is decreasing. For example, if $f(x) = x$, then

$$M(c) = \begin{cases} 1 & \text{if } c = 0, \\ 0 & \text{if } c > 0, \end{cases}$$

while if $f(x) = S(x, k)$ and $k > 0$, then

$$M(c) = \begin{cases} +\infty & \text{if } c < k, \\ 1 & \text{if } c = k, \\ 0 & \text{if } c > k. \end{cases}$$

We shall now show that if $c > 0$ and $M(c) < +\infty$, then $f(x) \leq S(x, c)$ for all $x \geq 0$. Since $f(x) \leq M(c)\, S(x, c)$, we have

$$f(x)^2 = \frac{1}{2} \int_0^{2x} f(t)\, dt \leq \frac{M(c)}{2} \int_0^{2x} S(t, c)\, dt = M(c) S(x, c)^2,$$

so that $f(x) \leq \sqrt{M(c)}\, S(x, c)$. Repeating this argument (infinitely often) gives $f(x) \leq S(x, c)$ for all x.

It is now clear that if $M(c) < +\infty$ for all $c > 0$, then $f(x) \leq S(x, c)$ for all $x \geq 0$, and all $c > 0$. Then, by letting $c \to 0$, we see that $f(x) = x$. Thus if f is *not* the function x, then there is some positive c such that $M(c) = +\infty$. Since $f(x)/S(x, c)$ is continuous on $[0, +\infty)$, this means that there is a sequence x_n in $[0, +\infty)$ such that $x_n \to +\infty$ and $f(x_n)/S(x_n, c) \to +\infty$. ■

Some remarks

In case the reader feels that the problem might, or perhaps should, admit an elementary solution (and it might), we remark that there is a vast literature on functional equations; see, for example, [**1**], [**5**], [**6**], [**7**], and [**8**]. Some of this applies to the functional equation

$$g(2x) = 2g(x)^2 - 1, \quad g(0) = 1, \tag{3}$$

which has solutions $\cos kx$ and $\cosh kx$. In fact, these are the only C^3-solutions in $\mathbb{R}$ [**6**, p. 406] and, better still, these are the only real solutions of (3) that are non-constant, continuous, even, and twice differentiable at $x = 0$ [**4**]. However, it is shown in [**3**] that if we do not require the existence of $g''(0)$, then there are other solutions of (3). In fact, the most general C^1-solution $g : \mathbb{R} \to [1, +\infty)$ of (3) *depends on an arbitrary function* [**6**, p. 406]. These remarks are about the functional equation (3). However, for C^1-functions $g : \mathbb{R} \to \mathbb{R}$, (3) is easily seen to be equivalent to the functional differential equation

$$g'(2x) = 2g(x)g'(x), \quad g(0) = 1, \tag{4}$$

so we find that there are uncountably many C^1-solutions of (4) other than $\cos kx$ and $\cosh kx$. This may or may not not contribute anything to the original problem, but it does suggest that the problem is not as simple as it might have appeared at first sight.

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Summary We discuss solutions of the equation $f(2x) = 2f(x)f'(x)$, which is essentially a delay differential equation, with the boundary condition $f'(0) = 1$, on the interval $[0, +\infty)$. In particular, we note that the only known solutions of this type that are positive when x is positive are the functions $c^{-1}\sinh(cx)$, where $c > 0$, and the function x.

Pi Day

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In 1932, Stanislaw Golab [**1**] showed that, for a large class of metrics defined in the plane—namely, the Minkowski metrics, which we define below—the perimeter of the unit disk ranges from 6 to 8. This means that the ratio of the perimeter to twice the radius, the quantity that corresponds to π in Euclidean space, can vary from 3 to 4 when we use these metrics.

We use the symbol PI for that ratio, so that we can allow it to vary with the metric we use while reserving the symbol π for its standard numerical value. In this note we construct metrics realizing all values of PI in the interval [3, 4], using methods easily understood by a student of linear algebra. We close by constructing a metric that makes PI equal 3.14 exactly. As it is, March 14 or 3/14 is only an approximate π day, but it is an exact PI day.

The metrics. A *metric* in the plane $\mathbb{R}^2$ is a function $D : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ such that

1. $D\big((a,b),(c,d)\big) = 0$ if and only if $a = c$ and $b = d$,
2. $D\big((a,b),(c,d)\big) = D\big((c,d),(a,b)\big)$, and
3. $D\big((a,b),(c,d)\big) \le D\big((a,b),(e,f)\big) + D\big((e,f),(c,d)\big)$

for all (a,b), (c,d), and (e,f) in $\mathbb{R}^2$. In this paper, we restrict our attention to *Minkowski metrics* [**2, 3**]. These are metrics that have two additional properties: For all a, b, c, d, x, y, and k real,

4. $D\big((a,b),(c,d)\big) = D\big((a+x, b+y),(c+x, d+y)\big)$, and
5. $D\big((0,0),(ka,kb)\big) = |k|\,D\big((0,0),(a,b)\big)$.

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These properties provide translation invariance and suitable rules for contraction and expansion. It follows that the unit disk centered at the origin in a Minkowski metric fully describes the metric. Further, it turns out that any set can be a unit disk in some Minkowski metric, provided it is centrally symmetric, convex, bounded, and has nonempty interior.

Our unit disks will be polygons and we'll get them with the max metric. The *max metric* is denoted D_M and is defined as

$$D_M\big((a,b),(c,d)\big) = \max\{|a-c|, |b-d|\}.$$

The unit disk (or unit ball) centered at the origin in this metric is

$$B_M = \{(a,b) : D_M\big((a,b),(0,0)\big) \le 1\}$$

and it has the shape of a square (FIGURE 1).

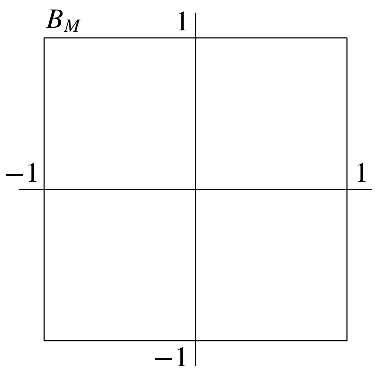

Figure 1 Unit disk for the max metric D_M

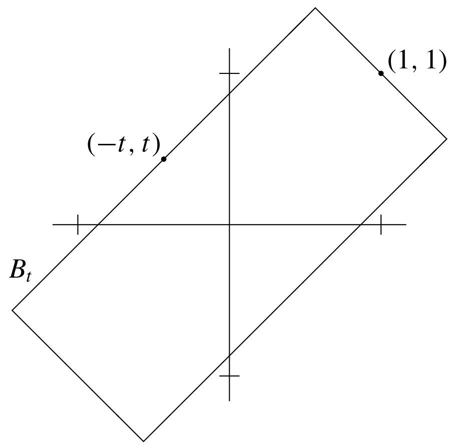

Figure 2 Unit disk for the metric D_t

The perimeter of the disk is 8. (That would be true whether we use the max metric or the Euclidean metric, but it is the perimeter measured by the max metric that determines PI.) The disk's radius is 1, so PI = 4 for this metric.

Variations of the max metric. A variation of the max metric [**4, 5**] that we will need involves a nonstandard basis for $\mathbb{R}^2$. Instead of the standard basis $(1,0)$, $(0,1)$, we will use a family of bases $\{(1,1), (-t,t) : 0 < t \le 1/2\}$. To define distance with one of the new bases, we express points in the plane as linear combinations of the new basis vectors and apply the max metric to these new coordinates.

Specifically, let (a,b) and (c,d) be points in the standard basis and let t be fixed. Write $(a,b) = a'(1,1) + b'(-t,t)$ and $(c,d) = c'(1,1) + d'(-t,t)$. Solving for a', b', c', and d', we get $a' = (a+b)/2$, $b' = (b-a)/2t$, $c' = (c+d)/2$, and $d' = (d-c)/2t$. Our metric in this new basis is $D_t\big((a,b),(c,d)\big) = \max\{|a'-c'|, |b'-d'|\}$, or, combining all these formulas,

$$D_t\big((a,b),(c,d)\big) = \max\left\{ \left|\frac{a+b}{2} - \frac{c+d}{2}\right|, \left|\frac{b-a}{2t} - \frac{d-c}{2t}\right| \right\}.$$

The unit disk in this metric is $B_t = \{(a,b) : D_t((0,0),(a,b)) \le 1\}$ and it has the shape of a rectangle (FIGURE 2).

If we were to use $(1/2, 1/2)$ and $(-1/2, 1/2)$ as our basis, then the max metric in this basis would be the taxicab metric, which is usually defined as $D((0, 0), (x, y)) = |x| + |y|$.

PI values in the interval [3, 4]. The metrics we need to illustrate Golab's theorem involve both D_M and D_t. Let $t \in (0, 1/2]$ be fixed and define $D^t\big((a, b), (c, d)\big) = \max\{D_M\big((a, b), (c, d)\big), D_t\big((a, b), (c, d)\big)\}$. It is not difficult to show that D^t is a metric and that the unit disk in this metric is the hexagon $B^t = B_M \cap B_t$ (FIGURE 3). The vertices are easily shown to be as labeled.

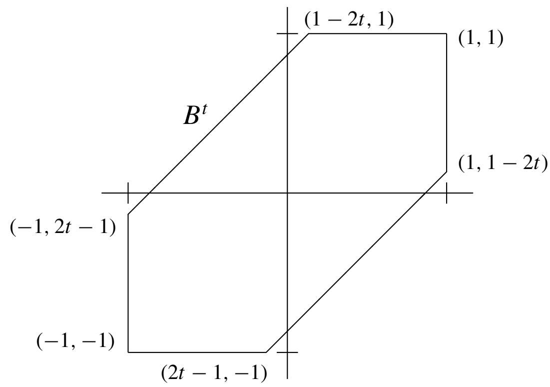

Figure 3 Unit disk for the metric D^t

To find the perimeter of B^t in the D^t metric, we have to find the lengths of the six sides in both D_M and D_t and take the larger of the two—then, add the six lengths. Starting with the segment joining $(1, 1 - 2t)$ and $(1, 1)$ and moving counterclockwise around the hexagon, the lengths in D_M are $2t$, $2t$, $2(1 - t)$, $2t$, $2t$, and $2(1 - t)$. The respective lengths in D_t are $1, 1, 2(1 - t), 1, 1, 2(1 - t)$. Hence, the lengths in D^t are $1, 1, 2(1 - t), 1, 1, 2(1 - t)$, so the perimeter is $8 - 4t$. Therefore, PI becomes $4 - 2t$ and since $t \in (0, 1/2]$, PI is seen to take values in $[3, 4)$.

Solving $4 - 2t = 3.14$, we get $t = .43$ exactly (the case illustrated in FIGURE 3). By using the metric $D^{.43}$, we can accurately call March 14 "Pi Day."

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Summary In 1932, Stanislaw Golab proved that, for a large class of metrics in the plane, the perimeter of the unit disk can vary from 6 to 8. Hence, the ratio corresponding to π can vary from 3 to 4. We illustrate this result by defining a family of metrics that can be understood easily by any student of linear algebra.

Proof Without Words: Fibonacci Triangles and Trapezoids

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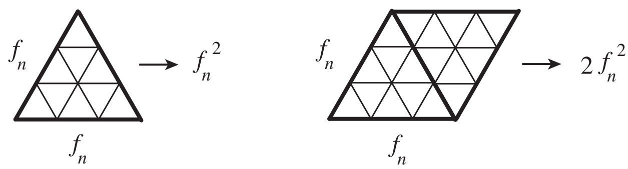
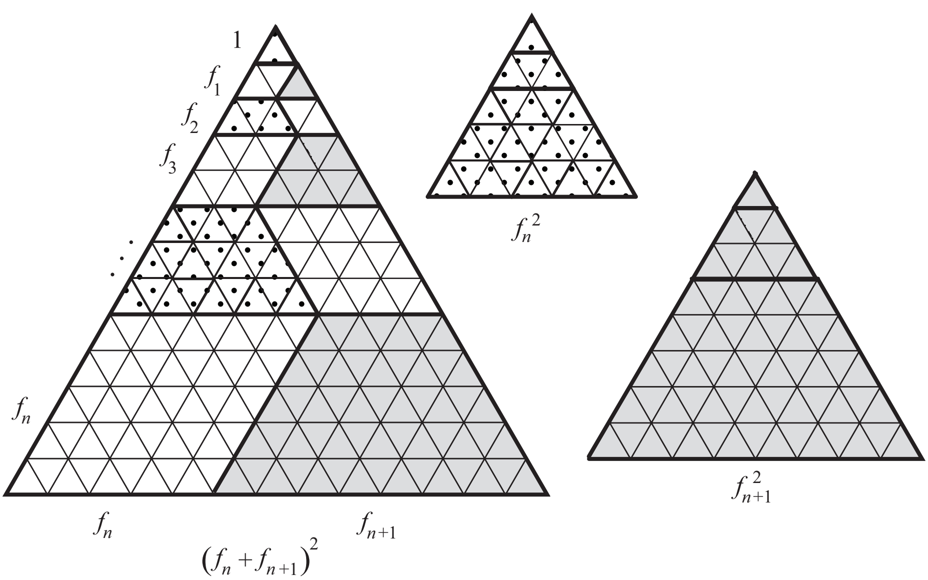
I. Counting triangles:



II. Identity: $$\sum_{k=1}^{n} 2f_k^2 + f_n^2 + f_{n+1}^2 = (f_n + f_{n+1})^2 .$$



The identity proved here reduces algebraically to $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$. This proof uses almost the same diagram as [**1**] to prove a different identity.

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The Scope of the Power Series Method

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A well-known way to find a solution of a differential equation is the *power series method*: assume that the solution has a convergent power series expansion, and use the equation to determine the coefficients, if possible. But when is it practical? What is the scope of the power series method? We show that (a) the scope of the method is much narrower than we might think, but (b) given the result (a), the scope is much broader than we might think—it takes in most "special functions."

A simple example for the use of the power series method is provided by the equation

$$U'(x) = U(x). \tag{1}$$

If $U(x)$ has series expansion $U(x) = \sum_{n=0}^{\infty} U_n\, x^n$, then term-by-term differentiation turns equation (1) into

$$\sum_{n=1}^{\infty} n\, U_n\, x^{n-1} = \sum_{n=0}^{\infty} U_n\, x^n.$$

Coefficients of like powers of x must be the same, so $nU_n = U_{n-1}$ for $n \geq 1$. Then

$$U_n = \frac{1}{n} \cdot U_{n-1} = \frac{1}{n} \cdot \frac{1}{n-1} \cdot U_{n-2} = \cdots = \frac{1}{n!} \cdot U_0,$$

and we conclude that $U(x) = \left(\sum_{n=0}^{\infty} x^n/n!\right)\, U_0 = e^x U_0$.

This procedure may seem cumbersome and unnecessary for first-order equations like (1). A standard example of a second-order equation for which there is no other obvious way to produce a solution is *Bessel's equation:*

$$x^2\, U''(x) + x\, U'(x) + [x^2 - \nu^2]\, U(x) = 0, \tag{2}$$

where ν is a constant. We look for a solution in the form

$$U(x) = x^{\nu} V(x) = x^{\nu} \sum_{n=0}^{\infty} V_n\, x^n.$$

Equation (2) for $U(x)$ is equivalent to the following equation for $V(x)$:

$$x\, V''(x) + (2\nu + 1)\, V'(x) + x\, V(x) = 0. \tag{3}$$

The reader is invited to determine a power series solution of (3) with constant term $V_0 = 1$ and show that it is unique, unless 2ν is a negative integer. (In fact, the reader who has taken a course in ordinary differential equations may already have been assigned this problem as an exercise.) The answer turns out to be $V_0 = 1$, $V_n = 0$ for odd n, and, for even $n \geq 2$,

$$V_n = \frac{1}{n(n+2\nu)} V_{n-2}.$$

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Second-order equations are of great importance in applications. The general linear homogeneous second-order equation is

$$p(x)\,U''(x) + q(x)\,U'(x) + r(x)\,U(x) = 0. \tag{4}$$

If we expect to use the power series method on such an equation, say in a neighborhood of $x = 0$, we need the coefficient functions themselves to have power series expansions in that neighborhood:

$$p(x) = \sum_{n=0}^{\infty} p_n\,x^n, \qquad q(x) = \sum_{n=0}^{\infty} q_n\,x^n, \qquad r(x) = \sum_{n=0}^{\infty} r_n\,x^n.$$

We look for a solution U of equation (4) with series expansion $U(x) = \sum_{n=0}^{\infty} U_n\,x^n$. Put this expansion, and the expansions of the coefficients, into equation (4). Collecting like powers of x leads to a formal expansion

$$\begin{aligned} 0 = {} & [2\,p_0\,U_2 + q_0\,U_1 + r_0\,U_0] \\ & + [2p_1\,U_2 + 6p_0\,U_3 + q_1\,U_1 + 2q_0\,U_2 + r_1\,U_0 + r_0\,U_1]\,x \\ & + [2p_2\,U_2 + 6p_1\,U_3 + 12p_0\,U_4 + q_2\,U_1 + 2q_1\,U_2 + 6q_0\,U_3 \\ & \quad + r_2\,U_0 + r_1\,U_1 + r_0\,U_2]\,x^2 + \cdots . \end{aligned}$$

In order for this to be correct for all x in a neighborhood of $x = 0$, the coefficient of each power x^n must vanish. This means that

$$2\,p_0\,U_2 + q_0\,U_1 + r_0\,U_0 = 0, \tag{5}$$

$$2p_1\,U_2 + 6p_0\,U_3 + q_1\,U_1 + 2q_0\,U_2 + r_1\,U_0 + r_0\,U_1 = 0, \tag{6}$$

$$\begin{aligned} 2p_2\,U_2 + 6p_1\,U_3 + 12p_0\,U_4 + q_2\,U_1 + 2q_1\,U_2 + 6q_0\,U_3 & \\ + r_2\,U_0 + r_1\,U_1 + r_0\,U_2 = 0, & \end{aligned} \tag{7}$$

and so on, for higher powers of x. The idea is to use these equations to determine the coefficients of U_n. For example, if $p_0 \neq 0$, then (5) determines U_2 from U_0 and U_1, (6) determines U_3 from U_0, U_1, and U_2, and so on—each coefficient can be calculated in turn, once U_0 and U_1 are specified. This process can be very tedious. As an example, consider the equation

$$U''(x) + e^x\,U(x) = 0.$$

Suppose that $U_0 = 1$ and $U_1 = 0$. What is U_5? What is U_{10}? Is this a practical method for determining the entire series?

The question we want to address here is: Under what conditions does this process of determining coefficients work in a simple, straightforward way? For a start, we take this to mean that the constant term (5) involves only the coefficients U_0 and U_1, the first degree term (6) involves only U_1 and U_2, and so on.

The question

Under what conditions on the coefficient functions p, q, r does the coefficient of x^n in the formal expansion of equation (4) involve only U_n and U_{n+1}?

The answer

Here we shall borrow (and add more detail to) an argument from the first chapter of [**1**].

One condition is immediate from expression (5): $p_0 = 0$. Expression (6) gives the condition $r_1 = 0$. Expression (7) gives two more conditions: $q_2 = r_2 = 0$. The reader is invited to examine the formal coefficient of x^3, and determine that at this stage there are three more conditions $p_3 = q_3 = r_3 = 0$. This points the way to the general answer.

THEOREM. *The equation for the coefficient of x^n in the formal power series expansion of equation* (4) *involves only the coefficients U_n and U_{n+1} if and only if the coefficient p is a polynomial of degree at most two without constant term, q is a polynomial of degree at most one, and r is constant.*

Proof. For general $n \geq 2$, the portion of the coefficient of x^n that involves U_0, U_1, and U_2 is

$$2p_n U_2 + q_n U_1 + r_n U_0.$$

To eliminate these terms when $n \geq 3$, we need $p_n = q_n = r_n = 0$ for all $n \geq 3$. Since we already know that p_0, q_2, r_2, and r_1 must vanish, this proves the necessity of the conditions. On the other hand, if the conditions are satisfied, then the coefficient of x^n is just

$$[n(n-1)p_2 + n\,q_1 + r_0]\,U_n + [(n+1)n\,p_1 + (n+1)q_0]\,U_{n+1},$$

so the conditions are sufficient. ■

This theorem is just the beginning of the story. As we shall see, essentially two interesting families of equations arise in this context.

We note first that if $p(x) \equiv 0$, then (4) is a first-order equation, so we shall assume that at least one of p_2 and p_1 is not zero. We may assume also that the leading coefficient of $p(x)$ is ± 1.

For particular values of the coefficients p_2, p_1, q_1, q_0, r_0, the coefficient of U_n or of U_{n+1} in (7) will vanish. Suppose that this does not happen. Then for each $n \geq 0$,

$$U_{n+1} = -\frac{n(n-1)p_2 + n\,q_1 + r_0}{(n+1)n\,p_1 + (n+1)q_0} \cdot U_n. \tag{8}$$

Then $U_0 = 0$ implies that all the coefficients are zero, while $U_0 \neq 0$ implies that none of the coefficients U_n is zero. We assume, therefore, that $U_0 \neq 0$.

We consider two cases, depending on whether $p_2 = 0$.

The hypergeometric equation

Suppose first that $p_2 \neq 0$. It follows from (8) that if $p_1 = 0$, then the formal series $\sum U_n x^n$ diverges for every $x \neq 0$ (ratio test). Therefore, in this case we must insist that $p_1 \neq 0$. We normalize by taking the leading coefficient p_2 to be -1. Then

$$p(x) = x(\alpha - x), \quad \alpha \neq 0.$$

A change of scale (taking αx as the new independent variable, relabeling it as x, and dividing the equation by α) reduces this to the form $p(x) = x(1-x)$. Constants a, b, c can then be chosen so that our equation (4) has the form

$$x(1-x)\,U'' + [c - (a+b+1)x]\,U'(x) - ab\,U(x) = 0. \tag{9}$$

This is known as the *hypergeometric equation.* More precisely, it is a family of equations indexed by the three parameters a, b, and c.

The confluent hypergeometric equation

Now consider the case $p_2 = 0$. We may take $p(x) = x$. Again we consider two cases, depending on whether $q_1 = 0$. Supposing first that $q_1 \neq 0$, we may change the scale and assume that $q_1 = -1$, so equation (4) has the form

$$xU''(x) + [c - x]\,U'(x) - a\,U(x) = 0. \tag{10}$$

This is known as the *confluent hypergeometric equation*, a family of equations indexed by the two parameters a and c.

A third equation

The remaining possibility is $p_2 = 0$ and $q_1 = 0$. Again we take $p_1 = 1$. If $r_0 = 0$, the resulting equation is simply $xU'' + q_0\,U' = 0$, a first-order equation for the derivative U'. If $r_0 \neq 0$, then after a change of scale we are left with an equation depending on a single parameter:

$$x\,U''(x) + c\,U'(x) + U(x) = 0. \tag{11}$$

As we shall see in the last section, this equation can be seen as simply a variant of the confluent hypergeometric equation.

We have reduced the interesting possibilities for a simple application of the power series method, after some normalizations, to just two cases: the hypergeometric equation (9) and the confluent hypergeometric equation (10). This would seem to be a severe limitation to the practical use of the method. Nevertheless, as we shall indicate in the last section, quite a number of interesting functions come up in this way, associated with equations (9) and (10).

The reader may have noticed that equation (3) does not appear to be covered by this discussion, even though we gave it as an example of the application of the power series method. A glance above reveals the reason for this exclusion: The sequence of equations for the coefficients of (3) links V_n with V_{n+2} rather than with V_{n+1}. But we disallowed such a linkage in our analysis of equation (4). We shall return to equation (3) in the next section, and again in the last section.

Another approach

The argument in the previous section can be organized in a more elegant and conceptual way. To do so, we use the "Euler derivative" $D = x\frac{d}{dx}$, i.e.,

$$DU(x) = x\,\frac{dU}{dx}(x) = x\,U'(x).$$

For this operator we have the identities

$$D\{x^n\} = nx^n, \quad D^2\{x^n\} = n^2x^n \tag{12}$$

and

$$D^2U(x) = x\,\frac{d}{dx}\left\{x\,\frac{dU}{dx}(x)\right\} = x^2\,U''(x) + x\,U'(x),$$

so that

$$x^2\,U''(x) = D^2U(x) - DU(x) = D(D-1)U(x). \tag{13}$$

As an example, let us multiply the confluent hypergeometric equation (10) by x and use the identity (13). The resulting equation can be written as

$$D(D+c-1)U(x) - x(D+a)U(x) = 0.$$

Now from the identity (12), we find that the coefficient of x^n is

$$n(c+n-1)U_n - (a+n-1)U_{n-1}.$$

If we set $U_0 = 1$ and determine U_n recursively, the result is the convergent series

$$U(x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(c)_n}\,x^n, \tag{14}$$

where the "extended factorials" $(a)_n$ and $(c)_n$ are defined by products

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1, 2, 3, \dots,$$

and similarly for $(c)_n$. These constructions are also called "rising powers." They can be expressed in terms of the gamma function $(a)_n = \Gamma(n+a)/\Gamma(a)$. (In deriving formula (14) we have tacitly assumed that c is not zero or a negative integer.)

The reader is invited to analyze the hypergeometric equation (9) and its solution in a similar way and determine a series solution that converges for $|x| < 1$.

Now let us take a second look at the question in the first section. If we multiply equation (4) by x^2 and use the identity (13), we can write equation (4) in the form

$$\widetilde{p}(x)D^2U(x) + \widetilde{q}(x)DU(x) + \widetilde{r}(x)U(x) = 0. \tag{15}$$

The identity (12) tells us that if we express the coefficient functions and the function U in equation (15) as power series and multiply out, then the coefficient of x^n is

$$\sum_{k=0}^{n} \left[(n-k)^2\widetilde{p}_k + (n-k)\widetilde{q}_k + \widetilde{r}_k\right] U_{n-k}. \tag{16}$$

In order for each such expression to involve only two consecutive coefficients from the expansion of U, the coefficients $\widetilde{p}_k, \widetilde{q}_k, \widetilde{r}_k$ must vanish, except for two consecutive values of k. After dividing out by some power of x (possibly x^0), we may assume that the allowed values of k are 0 and 1. This means that each of $\widetilde{p}$, $\widetilde{q}$, and $\widetilde{r}$ is a polynomial of degree at most 1. One more remark: At degree zero the sum (16) reduces to $\widetilde{r}_0\,U_0$. If $U_0 = 0$, then (except for some very special choices of coefficients) necessarily $U(x) \equiv 0$. Thus we assume that $\widetilde{r}_0 = 0$. This condition implies that every term in (15) is divisible by x. After dividing by x and collecting terms, (15) takes the form

$$[p_2x^2 + p_1x]\,U''(x) + [q_1x + q_0]\,U'(x) + r_0\,U(x) = 0.$$

Thus we have once again obtained the theorem above.

We leave it to the reader to use this approach to identify the general form of equations which, like equation (3) (which is related to equation (2)), link U_n and U_{n+2} rather than U_n and U_{n+1}, or, more generally, link U_n and U_{n+j} for some fixed j. (After some analysis, this reduces to equations (9) and (10) in the new variable $y = \alpha x^j$, for some choice of the scaling parameter α.)

Special functions

The results here show that use of the power series method to solve the second order equation (4) is simple and straightforward only under severe restrictions on the coefficient functions p, q, and r. Nevertheless, a broad territory remains. In fact, equations (9) and (10) account, after some transformations, for most of what are called "special functions."

The adjective "special" is used to distinguish these functions from the "elementary functions," although the two categories are not completely disjoint. Elementary functions are the functions seen in a first course in calculus: polynomials, the exponential function, the logarithm, the trigonometric functions, and functions that are constructed from these by algebraic operations and composition, such as

$$f(x) = \cos\left(\sqrt{3x^2 + 5e^x}\right).$$

Generally speaking, a special function is a function that is not an elementary function (apart from some special values of the parameters that index it), is useful enough to have been given a name and to have been studied intensively, and simple enough that many important properties and relations can be established. These include spherical functions, parabolic cylinder functions, functions associated with the names Legendre, Gauss, Kummer, Bessel, Hankel, Weber, Airy, Kelvin, and Whittaker, and polynomials associated with the names Legendre, Laguerre, Jacobi, Hermite, Chebyshev, and Gegenbauer. Each of the functions just named is a solution of an equation that is a variant of the hypergeometric equation or the confluent hypergeometric equation. (Some special functions that are not closely related to equations (9) and (10) are the gamma, beta, and zeta functions, and the various elliptic functions.)

Power series expansions are just one step in the study of such functions. For much more, browse classical treatises like [**4**] and [**5**]. For the most up-to-date and complete treatment, see [**3**], or the online version [**2**].

According to what we have just said, there should be some way of associating Bessel functions, which are solutions of equation (2), with one of the equations (9) or (10). We have associated solutions of Bessel's equation (2) with solutions of (3). Suppose that we write the function v in equation (3) as

$$V(x) = e^{-ix}\, W(2ix).$$

After some effort, one can see that equation (3) for V is equivalent to the following equation for the function $W(y)$:

$$y\, W''(y) - y\, yW'(y) + (2\nu + 1)\, W'(y) - \left(\nu + \frac{1}{2}\right)\, W(y) = 0.$$

This is just the confluent hypergeometric equation (10) with $c = 2\nu + 1 = 2a$. Thus, Bessel's equation can be regarded as a variant of a particular confluent hypergeometric equation.

As a final loose end, consider the equation (11). Does it contribute anything new? If we write the function U as

$$U(x) = V\left(\sqrt{2x}\right),$$

then equation (11) for U is equivalent to equation (3) for V. As we have seen, a further reduction leads to a special case of (10). Thus, in some sense, equation (10) accounts for equations (2), (3), and (11) as well.

The manipulations that led from (2) to (3) and from (3) and (11) to (10)—multiplication by a known function and change of variables—suggest how it is that so many different functions can be associated with the one pair of equations (9) and (10).

Another factor leading to this multiplicity of functions is the possibility of varying the parameters that index equations (9) and (10). For some values of the parameters the associated special function is an elementary function. Suppose that the parameter c in (9) or (10) is not zero or a negative integer. If $a = c$, the confluent hypergeometric function (14) is very familiar. If a is a negative integer, then (14) is a polynomial. The corresponding series in the hypergeometric case converges to the function $U(x) = (1-x)^{-b}$ if $a = c$, and is a polynomial if either of a or b is a negative integer. Many more such examples are given in the various references below, e.g., [**1**, sections 6.4, 8.1, and 8.7].

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Summary This note examines the application of the power series method to the solution of second order homogeneous linear ODEs, and shows that it works in a straightforward way only under restrictive conditions—in cases that reduce to the hypergeometric equation or the confluent hypergeometric equation. On the other hand, it is noted that these equations account for most "special functions."

PROBLEMS

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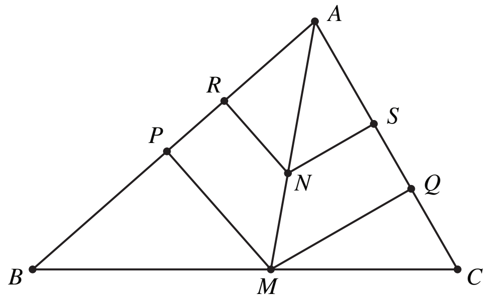
PROPOSALS

To be considered for publication, solutions should be received by July 1, 2013.

1911. *Proposed by Sadi Abu-Saymeh and Mowaffaq Hajja, Mathematics Department, Yarmouk University, Irbid, Jordan.*

Let ABC be a triangle and M a point on $\overline{BC}$ such that both angles $\angle BAM$ and $\angle MAC$ are acute. Let N be a point on $\overline{AM}$ and let P and R (respectively Q and S) be the feet of the perpendiculars from M and N onto $\overline{AB}$ (respectively $\overline{AC}$). Prove that if $\text{Area}(BMP) = \text{Area}(MCQ)$ and $\text{Area}(MNRP) = \text{Area}(NMQS)$, then M is the midpoint of $\overline{BC}$. Moreover, if $AB \neq AC$, then $\angle BAC = 90°$.



1912. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let $n \geq 3$ be an integer. Evaluate

$$\int_0^\infty \frac{x^n - 2x + 1}{x^{2n} - 1} dx.$$

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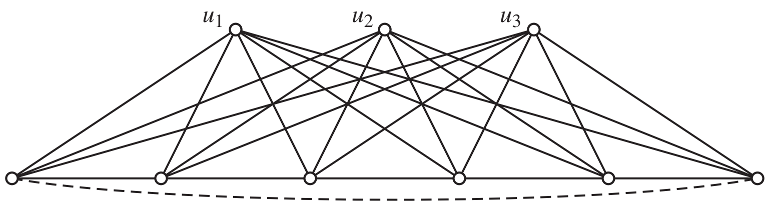
We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1913. *Proposed by Stan Wagon, Macalester College, St. Paul, MN.*

Let m and n be positive integers. The *m-suspension* of a graph G is the graph obtained by adding to G a set of m vertices $\{u_1, u_2, \ldots, u_m\}$ and all edges of the form vu_i, with $1 \leq i \leq m$ and v a vertex of G. The *fan graph $F_{m,n}$* is the m-suspension of an n-vertex path, while the *cone graph $C_{m,n}$* is the m-suspension of an n-cycle. The figure shows $F_{3,6}$ and, with the dashed edge, $C_{3,6}$. Let Δ denote a graph's maximum vertex degree.



(a) Determine all pairs (m, n) for which $F_{m,n}$ has a proper edge coloring using Δ colors (i.e., a coloring of the edges so that edges that share a vertex have different colors).

(b) Determine all pairs (m, n) for which $C_{m,n}$ has a proper edge coloring using Δ colors.

1914. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $h(0) = h'(0) = 0$, and suppose that h' is increasing on $\mathbb{R}$. Prove that if $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then

$$\int_0^1 [h(x) f(x) - x h(f(x))]\, dx \leq \int_0^1 x h(x)\, dx.$$

1915. *Proposed by Dietrich Trenkler, Department of Economics, University of Osnabrück, Osnabrück, Germany, and Götz Trenkler, Department of Statistics, University of Dortmund, Dortmund, Germany.*

Let $n = (n_1, n_2, n_3)^T \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi]$ a real number. In $\mathbb{R}^3$, a *proper rotation* by the angle θ about an axis in the direction of n can be carried out by the matrix

$$R = (\cos\theta) I + (1 - \cos\theta) n n^T + (\sin\theta) N,$$

where I denotes the identity matrix in $\mathbb{R}^3$ and

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}.$$

Determine the set

$$\mathfrak{F}(R) = \left\{ z^* R z : z \in \mathbb{C}^3 \text{ and } z^* z = 1 \right\},$$

where $z^* = \overline{z}^T$ denotes the conjugate transpose of z.

Quickies

Answers to the Quickies are on page 70. Quickies Q1025 and Q1026 are reprinted because their solutions were missing in the December 2012 issue.

Q1025. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Determine all real-valued continuous functions on $\mathbb{R}$ such that

(a) $f(x+y) = f(x) + f(y) + e^x e^y$ for all $x, y \in \mathbb{R}$,

(b) $f(x+y) = f(x) + f(y) + (e^x - 1)(e^y - 1)$ for all $x, y \in \mathbb{R}$.

Q1026. *Proposed by Tom Moore, Department of Mathematics and Computer Science, Bridgewater State University, Bridgewater, MA.*

Let R be a ring with unity $1_R \neq 0_R$. Can R have an odd number of idempotents?

Q1027. *Proposed by Jacob A. Siehler, Department of Mathematics, Washington & Lee University, Lexington, VA.*

Evaluate the double integral

$$\int_{-1}^{1}\int_{-1}^{1} \max(0, 1 + \min(1, ax + by))\, dy\, dx,$$

where a and b are arbitrary real constants.

Q1028. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $m > n \geq 1$ be integers. Prove that there is an nth power of a positive integer between any two mth powers of positive integers.

Solutions

Integer values of a binomial sum **February 2012**

1886. *Proposed by Jodi Gubernat and Tom Beatty, Florida Gulf Coast University, Fort Myers, FL.*

For which positive integers n is the function

$$f(n) = \sum_{k=\lfloor n/2 \rfloor}^{n} \left(1 - \frac{2k}{n}\right)^2 \binom{n}{k}$$

an integer?

Solution by Robert Calcaterra, University of Wisconsin-Platteville, Platteville, WI. Observe that $f(n) = \sum_{k=\lfloor n/2 \rfloor}^{n} (n^2 - 4kn + 4k^2)\binom{n}{k}/n^2$. Thus $f(n)$ is an integer if and only if

$$\sum_{k=\lfloor n/2 \rfloor}^{n-1} \frac{4k(n-k)}{n^2}\binom{n}{k} = \sum_{k=\lfloor n/2 \rfloor}^{n-1} \frac{4(n-1)}{n}\binom{n-2}{k-1}$$

is an integer. Therefore $f(n)$ is an integer if and only if

$$4\sum_{k=\lfloor n/2\rfloor}^{n-1}\binom{n-2}{k-1}\equiv 0 \pmod{n}.$$

Consider first the case in which n is even. Note that the terms in $\sum_{k=\lfloor n/2\rfloor}^{n-1}\binom{n-2}{k-1}$ are the terms from the midpoint on in the expansion of $(1+1)^{n-2}$. Hence

$$4\sum_{k=\lfloor n/2\rfloor}^{n-1}\binom{n-2}{k-1}=4\cdot\frac{1}{2}\cdot\left(2^{n-2}+\binom{n-2}{(n-2)/2}\right)=2^{n-1}+nC_{(n-2)/2},$$

where $C_k=\binom{2k}{k}/(k+1)$ denotes the kth Catalan Number. Therefore, $f(n)$ is an integer if and only if n is a divisor of 2^{n-1}. It follows that, for positive even integers n, $f(n)$ is an integer precisely when n is a power of 2.

Now suppose that n is odd. In this case, using the expansion of $(1+1)^{n-2}$, together with the fact that $\binom{n-2}{(n-3)/2}+\binom{n-2}{(n-1)/2}=\binom{n-1}{(n-1)/2}$, yields

$$4\sum_{k=\lfloor n/2\rfloor}^{n-1}\binom{n-2}{k-1}=4\cdot\frac{1}{2}\cdot\left(2^{n-2}+\binom{n-1}{(n-1)/2}\right)=2^{n-1}+(n+1)C_{(n-1)/2}.$$

Therefore, for positive odd integers n, $f(n)$ is an integer precisely when $2^{n-1}+C_{(n-1)/2}\equiv 0 \pmod{n}$. So, for example, $f(9)$ and $f(21)$ are integers. Furthermore, if n is an odd prime number, then $2^{n-1}\equiv 1 \pmod{n}$ and $C_{(n-1)/2}\equiv(n+1)C_{(n-1)/2}=2\binom{n-1}{(n-1)/2}\equiv 2(-1)^{(n-1)/2} \pmod{n}$. Hence, $f(n)$ is never an integer if n is an odd prime number.

Editor's Note. As most of our solvers realized, there was a typo on the problem. The lower index of the sum was meant to be $\lceil \frac{n}{2}\rceil$. In spite of this, it is an interesting and challenging problem to find a closed formula for the odd values of n for which $2^{n-1}+C_{(n-1)/2}\equiv 0 \pmod{n}$, or even to show that there are infinitely many. After 9 and 21, the next few values that work are 57, 237, 489, 669, 1461, 1641, 2181, 4377, 4449, 4593, 4881, 5097, 5853, 5997, 6393, and 6537. It is interesting to note that all of them have the form $n=3p$, where p is prime.

Also solved by George Apostolopoulos (Greece), Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Con Amore Problem Group (Denmark), Roger B. Eggleton (Australia), Michael Vowe (Switzerland), Haohao Wang and Jerzy Wojdyło, and the proposers.

The ellipse in all triangles with fixed H and O **February 2012**

1887. *Proposed by Elias Lampakis, Kiparissia, Greece.*

Given a circle $\mathcal{C}$ with center O and radius r, and a point H such that $0<OH<r$,

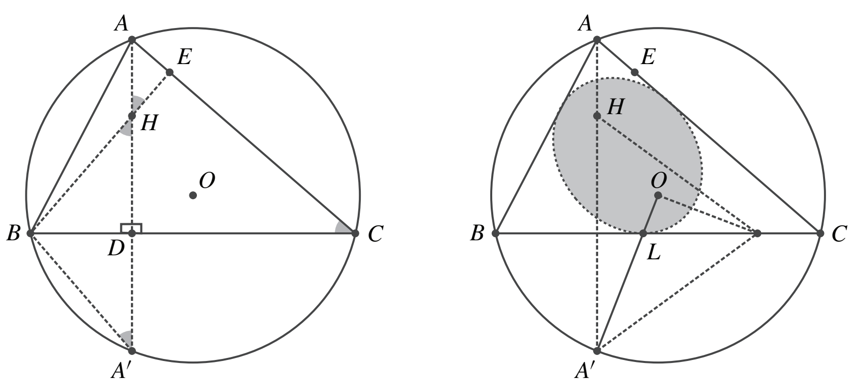
(a) Show that there are an infinite number of triangles inscribed in $\mathcal{C}$ with orthocenter H.

(b) Determine the set of points belonging to the interior of all triangles inscribed in $\mathcal{C}$ with orthocenter H.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Let A be an arbitrary point on $\mathcal{C}$ and let A' be the second point of intersection of the line AH with $\mathcal{C}$. Define B and C as the points of intersection of the perpendicular bisector of $\overline{HA'}$ with $\mathcal{C}$. Let D be the point of intersection of $\overline{AA'}$ and $\overline{BC}$, and let E be the point of intersection of the line BH and $\overline{AC}$. We have $\angle ECD = \angle ACB = \angle AA'B = \angle DHB$. Thus $\angle DHE = \pi - \angle DHB = \pi - \angle ECD$, and so $EHDC$ is cyclic. In particular, $\angle BEC = \angle ADC = \pi/2$. This shows that H is the orthocenter of $\triangle ABC$, and since there are infinitely many possibilities for point A, part (a) is proved.

Now for part (b), suppose that H is the orthocenter of a triangle ABC inscribed in circle $\mathcal{C}$. Because H is inside the circumcircle of $\triangle ABC$, it follows that $\triangle ABC$ is acute, and consequently O and H are inside the triangle. Let A' be the symmetric of H with respect to the line BC. It follows that $\angle AA'B = \angle A'HB$. In addition, $\angle A'HB = \angle ACB$ since $A'H \perp BC$ and $HB \perp AC$. It follows that $ABA'C$ is cyclic because $\angle AA'B = \angle ACB$, and thus A' belongs to $\mathcal{C}$. Let L be the intersection of $\overline{BC}$ and $\overline{OA'}$ and let X be an arbitrary point on $\overline{BC}$. Because $XH = XA'$, it follows by the triangle inequality that $OX + XH = OX + XA' \geq OA' = r$, with equality for $X = L$. Thus the side $\overline{BC}$ is tangent at L to the ellipse $\mathcal{E}$, defined by the locus of the points X, such that $OX + XH = r$. Similarly, the sides $\overline{AB}$ and $\overline{AC}$ are also tangent to the ellipse $\mathcal{E}$. Thus $\mathcal{E}$ is internally tangent to $\triangle ABC$.



Moreover, given a point M on the ellipse $\mathcal{E}$, let A' be the intersection of the ray OM and $\mathcal{C}$, and let A be the other intersection of the line $A'H$ and $\mathcal{C}$. Following the construction of part (a), we obtain a triangle ABC inscribed in $\mathcal{C}$, with orthocenter H, and such that $L = M$ is the intersection of $\overline{BC}$ and $\overline{OA'}$. Thus M is not in the interior of $\triangle ABC$. If N is a point outside $\mathcal{E}$, then there is point M in $\mathcal{E}$ closest to N. Repeating the preceding construction starting with this M, the line BC separates N from the interior of $\mathcal{E}$ and $\triangle ABC$ circumscribes $\mathcal{E}$. Thus N is not in the interior of $\triangle ABC$.

Therefore the set of points belonging to the interior of all triangles inscribed in $\mathcal{C}$ with orthocenter H is the interior of the region enclosed by ellipse $\mathcal{E}$.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Bruce S. Burdick, Roger B. Eggleton (part (a)), Eugene A. Herman, L. R. King, Ángel Plaza (Spain) and Javier Sánchez-Reyes (Spain), Raul A. Simón (part(a)) (Chile), Michael Vowe (Switzerland), John Zacharias, and the proposer.

Closures and complements **February 2012**

1888. *Proposed by Alex Aguado, Duke University, Durham, NC.*

Let $A \subseteq X$ be a subset of a topological space, and let $N(A)$ denote the number of sets obtained from A by alternately taking closures and complements (in any order). It is well known that $N(A)$ is at most 14. However, for exactly which $r \leq 14$ is it possible to find A and X such that $N(A) = r$?

Solution by Mark Bowron, Laughlin, NV.
The possible values of r are $\{1, 2, 4, 6, 8, 10, 12, 14\}$. The following sets A_r of real numbers satisfy $N(A_r) = r$ for $r \in \{2, 4, 6, 8, 10, 12, 14\}$: $A_2 = \varnothing$, $A_4 = (-\infty, 1)$, $A_6 = A_4 \cup (1, 2)$, $A_8 = A_6 \cup \{2\}$, $A_{10} = A_8 \cup \{3\}$, $A_{12} = A_{10} \cup [Q \cap (2, 3)]$, $A_{14} = A_{12} \cup \{4\}$. Since no subset of a nonempty topological space can equal its own complement, then $N(A)$ can be odd only if $A = X = \varnothing$. In this case, $N(A) = 1$.

Editor's Note. The fact that $N(A)$ is at most 14 is known as Kuratowski's Closure-Complement Problem.

Also solved by George Apostolopoulos (Greece), Bruce S. Burdick, Robert Calcaterra, Jaime Gutierrez, Victor Pambuccian, and the proposer.

Convergent sign patterns of the harmonic series **February 2012**

1889. *Proposed by Gary Gordon and Peter McGrath, Lafayette College, Easton, PA.*

For every positive integer k, consider the series

$$S_k = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}\right) - \left(\frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k}\right) + \cdots .$$

Thus $S_1 = \log 2$ and $S_2 = (\pi + 2\log 2)/4$.

(a) Prove that S_k converges for all k.

(b) Prove that

$$S_k = \int_0^1 \frac{x^k - 1}{(x^k + 1)(x - 1)}\, dx.$$

(c) Prove that the sequence $\{S_k\}$ is monotonically increasing and divergent.

Solution by Michel Bataille, Rouen, France.
For n a natural number, let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and for uniformity let $H_0 = 0$. The given series can be written as

$$S_k = \sum_{j=1}^{\infty} (-1)^{j+1} (H_{jk} - H_{(j-1)k}).$$

For $j \geq 1$, we have

$$H_{jk} - H_{(j-1)k} = \int_0^1 x^{(j-1)k} \Phi_k(x)\, dx,$$

where $\Phi_1(x) \equiv 1$, and if $k > 1$, $\Phi_k(x) = 1 + \sum_{j=1}^{k-1} x^j$, for all $x \in [0, 1]$. We observe that $\Phi_k(x) = (x^k - 1)/(x - 1)$ for all $x \neq 1$. We then compute the partial sums of S_k

in the following way:

$$\begin{aligned}\sum_{j=1}^{n}(-1)^{j+1}(H_{jk}-H_{(j-1)k}) &= \sum_{j=1}^{n}(-1)^{j+1}\int_0^1 x^{(j-1)k}\Phi_k(x)\,dx\\ &= \int_0^1 \Phi_k(x)\left(\sum_{j=1}^{n}(-1)^{j+1}x^{(j-1)k}\right)dx\\ &= \int_0^1 \Phi_k(x)\frac{1-(-x^k)^n}{1+x^k}\,dx\\ &= \int_0^1 \frac{\Phi_k(x)}{1+x^k}\,dx + (-1)^{n+1}\int_0^1 \Phi_k(x)\frac{x^{kn}}{1+x^k}\,dx.\end{aligned}$$

Because $0 \le \Phi_k(x) \le k$ and $x^k + 1 \ge 1$ for all $x \in [0, 1]$, it follows that

$$0 \le \int_0^1 \Phi_k(x)\frac{x^{kn}}{1+x^k}\,dx \le \int_0^1 kx^{kn}\,dx = \frac{k}{kn+1}.$$

Thus the squeeze principle implies that

$$\lim_{n\to\infty}\int_0^1 \Phi_k(x)\frac{x^{kn}}{1+x^k}\,dx = 0.$$

This shows parts (a) and (b) of the problem. For part (c), we use part (b) and observe that, for all $j \ge 1$,

$$S_{j+1} - S_j = \int_0^1 \frac{2x^j}{(x^j+1)(x^{j+1}+1)}\,dx > 0.$$

This shows that $\{S_k\}$ is an increasing sequence. Moreover,

$$S_{j+1} - S_j = \int_0^1 \frac{2x^j}{(x^j+1)(x^{j+1}+1)}\,dx \ge \int_0^1 \frac{x^j}{x^{j+1}+1}\,dx = \frac{1}{j+1}\ln 2.$$

Therefore, for all $k \ge 2$,

$$S_k = S_1 + \sum_{j=1}^{k-1}(S_{j+1}-S_j) \ge \ln 2 + (H_k - 1)\ln 2 = H_k \ln 2,$$

which implies that $\{S_k\}$ is divergent because the harmonic series is divergent.

Editor's Note. A few solutions were based on the Alternating Series Test and Abel's Theorem about the continuity of the power series at an endpoint of the interval of convergence. For part (c), some submissions used Lebesgue's Monotone Convergence Theorem. Traian Viteam has pointed out that the solution of Problem 11499 in *Amer. Math. Monthly* **119** (March 2012) p. 254, shows that for every $n \in \mathbb{N}$,

$$S_n = \frac{\ln 2}{n} + \frac{\pi}{n^2}\sum_{j=1}^{\lfloor n/2\rfloor}(n+1-2j)\cot\frac{(2j-1)\pi}{2n}.$$

In particular, one obtains that $S_1 = \ln 2$, $S_2 = \frac{1}{2}\ln 2 + \frac{1}{4}\pi$, $S_3 = \frac{1}{3}\ln 2 + \frac{2}{9}\pi\sqrt{3}$, $S_4 = \frac{1}{4}\ln 2 + \frac{1}{4}\pi\sqrt{2} + \frac{1}{8}\pi$, etc. Problem 11499 was proposed by Omran Kouba, who has shown that the following asymptotic formula is true:

$$S_n = \ln\left(\frac{2n}{\pi}\right) + \gamma + \frac{\ln 2}{n} - \frac{\pi^2}{72n^2} + O\left(\frac{1}{n^4}\right).$$

Also solved by Michael Andreoli, George Apostolopulous (Greece), Robert Calcaterra, Enkel Hysnelaj (Australia) and Elton Bojaxhiu (Germany), Bruce S. Burdick, Robert L. Doucette, Roger B. Eggleton (Australia), Michael Goldenberg and Mark Kaplan, William R. Green, Eugene A. Herman, Omran Kouba (Syria), Charles Lindsey, Jody M. Lockhart, David J. Lowry, Bob Mallison (IN), Northwestern University Math Problem Solving Group, Jose M. Pacheco (Spain) and Ángel Plaza (Spain), Paolo Perfetti (Italy), Tomas Persson (Sweden) and Mikael P. Sundqvist (Sweden), Joel Schlosberg, Nicholas C. Singer, Texas State University Problem Solvers Group, Traian Viteam, and the proposers. There were 11 incomplete or incorrect solutions.

Writing *m* as a sum of a square and a prime modulo *n* **February 2012**

1890. *Proposed by Erwin Just (Emeritus), Bronx Community College of the City University of New York, Bronx, NY.*

Let m and n be positive integers. Prove that there exist an integer k and a prime p such that $m \equiv k^2 + p \pmod{n}$.

I. *Solution by John H. Smith, Needham, MA.*
Let k be the product of the primes dividing n but not m ($k = 1$ if there are none). If a prime p divides n and m, then it does not divide k, and hence does not divide $m - k^2$; if p divides n but not m, then it divides k, and hence does not divide $m - k^2$. Thus n and $m - k^2$ are relatively prime. So, by Dirichlet's Theorem, the arithmetic progression $\{m - k^2 + in : i \in \mathbb{N}\}$ contains infinitely many primes. Any one of these will satisfy the congruence of the problem.

Note that the same argument works if k^2 is replaced by any positive power of k.

II. *Solution by Bruce S. Burdick, Department of Mathematics, Roger Williams University, Bristol, RI.*
According to problem 1872, whose solution appeared in this *Magazine* **85** (June 2012) p. 231, given an integer $n > 1$, any integer m may be written as $m \equiv a + b \pmod{n}$, where a is an integer that is relatively prime to n and b is an integer such that $b^2 \equiv b \pmod{n}$. Having chosen such a and b, we apply Dirichlet's Theorem to assert that there is a prime p such that $p \equiv a \pmod{n}$. We then have $m \equiv a + b \equiv p + b^2 \pmod{n}$.

Also solved by George Apostolopoulos (Greece), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Lindsay N. Childs, John Christopher, Roger B. Eggleton (Australia), Tom Jager, Omran Kouba (Syria), S. C. Locke, R. Keith Roop-Eckart, Joel Schlosberg, Nicholas C. Singer, Texas State University Problem Solvers Group, Traian Viteam (Uruguay), and the proposer.

Answers

Solutions to the Quickies from page 65.

A1025. (a) There are no functions, continuous or not, that satisfy

$$f(x+y) = f(x) + f(y) + e^x e^y.$$

To see this, note that $y = 0$ in the above equation gives $e^x = -f(0)$ for all real x, which is a contradiction.

(b) Suppose that f is a continuous solution of

$$f(x+y) = f(x) + f(y) + (e^x - 1)(e^y - 1). \qquad (1)$$

Let $g(x) = f(x) - e^x + 1$. Note that g is a continuous function on $\mathbb{R}$. Substituting $f(x) = g(x) + e^x - 1$ into (1) gives

$$g(x + y) = g(x) + g(y).$$

Since g is continuous, it is well known that $g(x) = Ax$ for some constant A. Thus any continuous solution of (1) must be of the form $f(x) = Ax + e^x - 1$, where A is an arbitrary constant. Furthermore, it is easy to verify that $f(x) = Ax + e^x - 1$ is a solution of (1) for every $A \in \mathbb{R}$. Therefore, these are all the continuous solutions.

A1026. R cannot have an odd number of idempotents. Let e be an idempotent of R. Note that $(1_R - e)(1_R - e) = 1_R - e - e + e^2 = 1_R - e$, so $1_R - e$ is also an idempotent. If $1_R - e = e$, then multiplying through by e gives $e - e^2 = e$, and so $0_R = e$. But then our supposition yields $0_R = e = 1_R - e = 1_R - 0_R = 1_R$, which is a contradiction. Therefore, $1_R - e \neq e$ and thus all of the idempotent elements in R may be paired up, so R has an even number of them.

A1027. The integrand once unraveled is equal to

$$\begin{cases} 0, & \text{if } ax + by \leq -1, \\ (ax + by) - (-1), & \text{if } -1 \leq ax + by \leq 1, \text{ and} \\ 2, & \text{if } ax + by > 1. \end{cases}$$

Thus the integral describes the volume of the region inside the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$ and below the plane with equation $z = ax + by$. Because the plane passes through the origin and the cube is centrally symmetric, the integral is equal to half the total volume of the cube. That is, the answer is 4, independently of the values of a and b.

A1028. It is enough to show that for every positive integer a, there is an integer b such that $a^m < b^n < (a + 1)^m$.

Let $f(x) = x^{m/n}$. Because $m/n > 1$, it follows that f is strictly convex on $(0, \infty)$. Thus by Jensen's Inequality, $\frac{1}{2}(f(c + 1) - f(c - 1)) > f(c)$ for every $c \geq 1$; that is, $f(c + 1) - f(c) > f(c) - f(c - 1)$ for every $c \geq 1$. Therefore, $(a + 1)^{m/n} - a^{m/n} = f(a + 1) - f(a) > f(a) - f(a - 1) > \cdots > f(2) - f(1) > f(1) - f(0) = 1$. Thus, there is an integer b such that $a^{m/n} < b < (a + 1)^{m/n}$. Raising this inequality to the exponent n gives the desired power.

Corrections and Omissions

The following names were inadvertently or incorrectly omitted from the lists of readers who sent correct solutions: Jack Abad, Paul Abad, and Victor Abad, problem 1854; Michel Bataille, problem 1881; Dmitry Fleischmann, problems 1854, 1856, and 1862; Marty Getz and Dixon Jones, problem 1864; Michael Goldenberg and Mark Kaplan, problems 1851, 1855, 1860, and 1864; Eugene A. Herman, problems 1856, 1861, 1862, and 1863; and Elias Lampakis, problem 1858. Despite the delay, we are glad to recognize these solvers' contributions.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Hartnett, Kevin, An ABC proof too tough even for mathematicians, *Boston Globe* (4 November 2012), http://www.bostonglobe.com/ideas/2012/11/03/abc-proof-too-tough-even-for-mathematicians/o9bja4kwPuXhDeDb2Ana2K/story.html; Math even mathematicians don't understand, (11 December 2012), http://bostonglobe.com/news/science/2012/12/11/math-even-mathematicians-don-understand/e41V2lixnAVyB63X4NTPaO/story.html.

Andrew Wiles secluded himself for most of a decade, proved Fermat's Last Theorem, and is now Sir Andrew Wiles; Grigori Perlman did likewise, is acknowledged to have proved the Poincaré conjecture, and refused the world's rewards and acclaim. So what about Shinichi Mochizuki, who followed a similar path, claimed in August to have proved the ABC conjecture, and declines interviews? A problem is that he is the "sole practitioner" of a branch of mathematics that he invented to attack the problem. Is a proof a proof if no one but the author understands it? Author Hartnett reminds us of Henry Pogorzelski, who has worked on the Goldbach conjecture for decades and announced successive "improved" proofs of it. Hartnett: "[N]o mathematicians can understand it or are even willing to invest the time to try." Hartnett concludes with a "ruthless truth": Mathematicians who make big claims are "obligated both to be right and to make themselves understood."

Havil, Julian, *The Irrationals: A Story of the Numbers You Can't Count On*, Princeton University Press, 2012; x + 294 pp, $29.95. ISBN 978-0-691-14342-2.

Instead of defining irrational numbers by what they are not, the author defines them as real numbers "having different distances from all rational numbers." He details their rise in ancient Greece, follows them through the Middle Ages, embarks on continued fraction representations, treats the discovery of algebraic irrationals, investigates the zeta function, pauses over transcendentals, and concludes by asking if irrationality matters (and for what). The book is intended for both those fluent in calculus and others "whose curiosity and enthusiasm are great." But "The informed reader may be disappointed by the omission of some material ... by design and undoubtedly there is much more that is missing by accident." Or included by accident: The details of how to find Roger Apéry's tomb seem unnecessary, but the author is to be congratulated on the fruits of the search: a photograph of the covering stone with its statement that $\zeta(3) \neq p/q$.

Lucianovic, Stephanie V.W., Saddle up for maximum satisfaction (mathematically speaking), CNN.com Blogs, http://eatocracy.cnn.com/2012/05/21/best-snack-shape/ .

Here's an opportunity to add to your teaching aids for teaching about critical points in multivariable calculus while at the same time nourishing your students. The wife of a mathematician discourses on his comparison of the relative value of the shapes of two national brands of stackable potato chips: one a parabolic cylinder (an indefinite quadratic form) and the other the (inequivalent) saddle shape of a hyperbolic paraboloid (a positive semidefinite quadratic form).

Math. Mag. **86** (2013) 72–73. doi:10.4169/math.mag.86.1.072.

Stewart, Ian, *Visions of Infinity: The Great Mathematical Problems*, Basic Books; viii + 311 pp, \$26.99. ISBN 978–0–465–02240–3.

This superb collection focuses on unsolved problems plus problems that have been solved in the last 50 years. All your favorites are here (Goldbach, four-color, Kepler, Mordell, Fermat, Riemann, etc.), including every one of the Millennium Prize problems; and all are explained with scarcely an equation but with plenty of intuition, understanding, and context. In addition to the 14 problems explained in detail, Stewart offers sketches of another dozen unsolved problems (admittedly chosen in part because of ease of understanding them).

Brilleslyper, Michael A., et al., *Explorations in Complex Analysis*, Mathematical Association of America, 2012; xviii + 373 pp, \$60 (MAA member: \$48). ISBN 978-0-88385-778-67.

In an undergraduate course in complex analysis, one can spend a lot of time extending functions from the reals to complex numbers and dealing with representations by series. This book shows how to make the course far more exciting by delving into current research topics: complex dynamics (how does Newton's method behave?), minimal surfaces in R^3, flows, anamorphic images, one-to-one maps between regions, and circle packings (make your own!). What keeps up the excitement is the plentitude of figures, the generous use of color, and the Java applets provided by the authors. There are exercises, explorations, and projects of varied magnitude.

Körner, T.W., *Naive Decision Making: Mathematics Applied to the Social World*, Cambridge University Press, 2008; xv + 373 pp, \$44 (P). ISBN 978-0-521-73163-8.

Glomski, Matthew, and Edward Ohanian, Eradicating a disease: Lessons from mathematical epidemiology, *College Mathematics Journal* 43 (2) (March 2012) 123–132.

This book by Körner is hyped by the publisher as helping you to choose the best restaurant, or whom to marry, or how to make money gambling. In truth, as the author makes clear in the Introduction, it requires a year of calculus ("the mathematics is the message") and "prospective readers should expect to learn more about mathematics than about decision making." I would add that the reader needs a stomach for mathematical notation and proofs, as well as an appetite for the fine points of probability. The rewards of the journey are many. Particularly enlightening is the summary of Daniel Bernoulli's use of Edmund Halley's life tables to consider the balance between the risk of dying from smallpox vs. the risk of dying from inoculation against it. (A video of Körner lecturing on this topic is at http://www.gresham.ac.uk/lectures-and-events/mathematics-and-smallpox; and further details, analysis, and application to other diseases are in the article by Glomski and Ohanian.) Sorting, shuffling, shortest paths, voting, game theory, and gambling all appear, sometimes in novel contexts or with surprising results. At last, here is a book on real-life topics at a level beyond the popular, a level at which mathematics students can see in action what they have learned.

Klarreich, Erica, An uncertain prognosis for Medicare auctions: Part I, *SIAM News* 45 (4) (May 2012) 3, 7; Part II, 45 (5) (June 2012) 8, 6.

"If you want to design an auction, you have to consult an auction expert." Apparently not, if you are the federal Centers for Medicare and Medicaid Services (CMS), which was charged with implementing competitive bidding for medical equipment. A mathematician and two economists have analyzed the CMS "median-price auction with opt-out," in which winning bidders are paid the median of the winning bids but can opt out of supplying if they don't like the price. Maybe you are already sensing what could go wrong.... The researchers tested such an auction on volunteers: Disaster—failure to procure the desired amount of supplies—predominated. More than three years after the CMS auctions, CMS has not revealed the results.

Linebaugh, Kate, The ups and downs of making elevators go: Creating: Theresa Christy, elevator engineer, *Wall Street Journal* (1–2 December 2012) C11, http://online.wsj.com/article/SB10001424127887324469304578143200385871618.html?KEYWORDS=elevators .

A mathematics major has a unique job: optimizing elevator algorithms. To become a manager, she got an M.B.A.—but then realized: "I really like solving the puzzles myself. I didn't like assigning them to other people. I was a little jealous." That's the spirit of mathematical curiosity!

NEWS AND LETTERS

73rd Annual William Lowell Putnam Mathematical Competition

Editor's Note: Additional solutions will be printed in the *Monthly* later in the year.

PROBLEMS

A1. Let $d_1, d_2, \ldots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

A2. Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

A3. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function such that

(i) $f(x) = \dfrac{2 - x^2}{2} f\left(\dfrac{x^2}{2 - x^2}\right)$ for every x in $[-1, 1]$,

(ii) $f(0) = 1$, and

(iii) $\lim\limits_{x \to 1^-} \dfrac{f(x)}{\sqrt{1 - x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

A4. Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

A5. Let $\mathbb{F}_p$ denote the field of integers modulo a prime p, and let n be a positive integer. Let v be a fixed vector in $\mathbb{F}_p^n$, let M be an $n \times n$ matrix with entries in $\mathbb{F}_p$, and define $G : \mathbb{F}_p^n \to \mathbb{F}_p^n$ by $G(x) = v + Mx$. Let $G^{(k)}$ denote the k-fold composition of G with itself, that is, $G^{(1)}(x) = G(x)$ and $G^{(k+1)}(x) = G(G^{(k)}(x))$. Determine all pairs p, n for which there exist v and M such that the p^n vectors $G^{(k)}(0)$, $k = 1, 2, \ldots, p^n$, are distinct.

A6. Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically 0?

Math. Mag. **86** (2013) 74–80. doi:10.4169/math.mag.86.1.074.

B1. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, then the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

B2. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property: If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.

B3. A round-robin tournament among $2n$ teams lasted for $2n - 1$ days, as follows. On each day, every team played one game against another team, with one team winning and one team losing in each of the n games. Over the course of the tournament, each team played every other team exactly once. Can one necessarily choose one winning team from each day without choosing any team more than once?

B4. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

B5. Prove that, for any two bounded functions $g_1, g_2 : \mathbb{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$,

$$\sup_{s\in\mathbb{R}} \big(g_1(s)^x g_2(s)\big) = \max_{t\in\mathbb{R}} \big(xh_1(t) + h_2(t)\big).$$

B6. Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

SOLUTIONS

Solution to A1. Without loss of generality, assume that the d_i are in nondecreasing order. We then need $d_{i+2}^2 < d_{i+1}^2 + d_i^2$ for some i. If $d_3^2 \geq d_1^2 + d_2^2$, then $d_3^2 \geq 2d_1^2$. If in addition $d_4^2 \geq d_3^2 + d_2^2$, then $d_4^2 \geq 3d_1^2 = F_4 d_1^2$, where F_i denotes the ith Fibonacci number. By induction, either we succeed, or $d_i^2 \geq F_i d_1^2$. But $F_{12} = 144$, $d_{12} < 12$, and $d_1 > 1$, so we must succeed at some point.

Solution to A2. Assume that $a * c = b * c$, and let $e_a, d \in S$ satisfy $a * e_a = a$ and $c * d = e_a$. Then

$$a = a * e_a = a * (c * d) = (a * c) * d = (b * c) * d = b * (c * d) = b * e_a.$$

Repeating the steps so far with a and b interchanged, there exists $e_b \in S$ such that $a * e_b = b * e_b = b$. Therefore,

$$\begin{aligned} a = b * e_a = (a * e_b) * e_a &= a * (e_b * e_a) \\ &= a * (e_a * e_b) = (a * e_a) * e_b = a * e_b = b. \end{aligned}$$

Solution to A3. $f(x) = \sqrt{1-x^2}$.
Proof: On $(-1, 1)$, set $g(x) = f(x)/\sqrt{1-x^2}$. Then

$$\begin{aligned} g(x) &= \frac{(2-x^2)/2}{\sqrt{1-x^2}}\, f\left(\frac{x^2}{2-x^2}\right) \\ &= \frac{2-x^2}{2\sqrt{1-x^2}}\sqrt{1-\left(\frac{x^2}{2-x^2}\right)^2}\, g\left(\frac{x^2}{2-x^2}\right) \\ &= g\left(\frac{x^2}{2-x^2}\right). \end{aligned}$$

On $(-1, 1)$, $x^2/(2-x^2) \leq |x|$ with equality only for $x = 0$, so the sequence

$$x,\ \frac{x^2}{2-x^2},\ \frac{\left(\frac{x^2}{2-x^2}\right)^2}{2-\left(\frac{x^2}{2-x^2}\right)^2}, \ldots$$

always has limit 0. Thus, by continuity of g, $g(x) = g(0) = 1$ for all x. It follows that $f(x) = \sqrt{1-x^2}$, where the continuity of f was used to show equality at the endpoints.

Note. As the proof shows, condition (iii) is actually unnecessary. (It was left in to provide a hint of the form of the solution.)

Solution to A4. Let $a_1 < a_2 < a_3$ be consecutive terms in S. We need only show $a_2 - a_1 = a_3 - a_2$. If not, replacing A and r with $-A$ and $-r$ if necessary, we may assume $a_2 - a_1 < a_3 - a_2$. Let $b_k \in B$ such that $ra_k \equiv b_k \pmod q$, $k = 1, 2, 3$. Replacing r and B with $-r$ and $-B$ if necessary, we may assume $b_1 \leq b_2$. We have

$$(a_3 - a_2)(b_2 - b_1) \equiv r(a_3 - a_2)(a_2 - a_1) \equiv (a_2 - a_1)(b_3 - b_2) \pmod q.$$

Because

$$\begin{aligned} |(a_3 - a_2)(b_2 - b_1) - (a_2 - a_1)(b_3 - b_2)| &\leq (a_3 - a_2)|b_2 - b_1| + (a_2 - a_1)|b_3 - b_2| \\ &\leq (a_3 - a_1) \cdot |B| \leq |A| \cdot |B| < q\,, \end{aligned}$$

we have $(a_3 - a_2)(b_2 - b_1) = (a_2 - a_1)(b_3 - b_2)$ or $\dfrac{b_2 - b_1}{a_2 - a_1} = \dfrac{b_3 - b_2}{a_3 - a_2} \geq 0$, so $a_2 - a_1 < a_3 - a_2$ implies $b_2 - b_1 \leq b_3 - b_2$. Then, however,

$$a_2 < 2a_2 - a_1 < a_3, \quad r(2a_2 - a_1) \equiv 2b_2 - b_1 \pmod q,$$

and

$$b_2 \leq 2b_2 - b_1 \leq b_3,$$

so $2b_2 - b_1$ is in the interval B and hence $2a_2 - a_1 \in S$, a contradiction.

Solution to A5. Such v and M exist for $n = 1$ and all p and for $n = 2$, $p = 2$.

For $n = 1$, set $v = [1]$ and $M = [1]$. For $p = n = 2$, set

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Conversely, suppose v and M exist. First observe that to get distinct values, the earliest possible occurrence of 0 is $G^{(p^n)}(0)$. Thus,

$$v + Mv + M^2v + \cdots + M^{p^n-1}v = 0.$$

Multiplying by M and combining the two expressions yields $M^{p^n}v = v$. But then, for all k,

$$M^{p^n}(v + Mv + \cdots + M^k v) = v + Mv + \cdots + M^k v.$$

Thus, M^{p^n} is the identity matrix. It follows that the minimal polynomial of M divides $x^{p^n} - 1 = (x-1)^{p^n}$. By the Cayley-Hamilton theorem, the minimal polynomial of M divides its characteristic polynomial; in particular, the minimal polynomial has degree at most n and so $(M - I)^n = 0$.

If neither $n = 1$ nor $p = n = 2$ holds, $p^{n-1} - 1 \geq n$, so $(M - I)^{p^{n-1}-1} = 0$. However,

$$(x-1)^{p^{n-1}-1} = \frac{(x-1)^{p^{n-1}}}{x-1} = \frac{x^{p^{n-1}} - 1}{x-1} = 1 + x + \cdots + x^{p^{n-1}-1}.$$

But then $G^{(p^{n-1})}(0) = 0$, a contradiction.

Solution to A6. Yes, $f(x, y)$ is identically 0, even if one only considers rectangles with sides parallel to the x- and y- axes.

For every $w > 0$ and every x, y,

$$\int_x^{x+w} \int_y^{y+1/w} f(u, v)\, dv\, du = 0.$$

Differentiating with respect to x, by the fundamental theorem of calculus we have

$$\int_y^{y+1/w} [f(x+w, v) - f(x, v)]\, dv = 0.$$

Differentiating with respect to y this time yields

$$f(x+w, y+1/w) - f(x, y+1/w) - f(x+w, y) + f(x, y) = 0.$$

Therefore, $f(x+w, y) - f(x, y)$ has period $1/w$ in y. Thus,

$$f(x+w+z, y) - f(x+z, y) - f(x+w, y) + f(x, y) \tag{1}$$

has periods $1/w$ and $1/z$ in y. Choosing w and z so that z/w is irrational, this and continuity imply (1) is independent of y, because any real y can be approximated arbitrarily closely by numbers of the form $m/w + n/z$ with m, n integers. Integrating (1) over any rectangle of the form $[x, x+\varepsilon] \times [y, y + 1/\varepsilon]$ gives four terms that are each the integral of $f(x, y)$ over some shifted rectangle of area 1, so it yields 0. By taking ε sufficiently small, continuity implies that (1) is identically 0. It follows that $f(x+w, y) - f(x, y)$ has period z in x. Since we may choose any z for which z/w is irrational, $f(x+w, y) - f(x, y)$ is independent of x. As above, integrating over $[x, x+1/\varepsilon] \times [y, y+\varepsilon]$, it follows that it is identically 0. Since w is arbitrary $f(x, y)$ is independent of x. Similarly it is independent of y, hence constant, hence identically 0.

Solution to B1. By rule (ii), $f_2(f(x)) = \ln(f(x)+1) \in S$ and $\ln(g(x)+1) \in S$, so that

$$\ln(f(x)+1) + \ln(g(x)+1) = \ln\big(f(x)g(x) + f(x) + g(x) + 1\big) \in S.$$

Therefore,

$$e^{\ln\big(f(x)g(x)+f(x)+g(x)+1\big)} - 1 = f(x)g(x) + f(x) + g(x) \in S.$$

Because $f(x) + g(x) \in S$ and $f(x)g(x) + f(x) + g(x) \geq f(x) + g(x)$ for every $x \in [0, \infty)$, it follows that $f(x)g(x) \in S$.

Solution to B2. Let $F_1, \dots, F_f$ be the faces of P, and let $\{B_1, \dots, B_n\}$ be a collection of n balls of radii $r_1, \dots, r_n$ respectively, such that $\cup_{i=1}^f F_i \subseteq \cup_{j=1}^n B_j$. Denote by $A(X)$ the area of a two-dimensional figure X, and by A the total surface area of P; note that

$$A = \sum_{i=1}^{f} A(F_i), \quad V = \frac{4}{3}\pi \sum_{j=1}^{n} r_j^3.$$

Since

$$A(F_i \cap B_j) \leq \pi r_j^2,$$

it follows that

$$A = \sum_{i=1}^{f} A(F_i) \leq \sum_{i=1}^{f}\sum_{j=1}^{n} A(F_i \cap B_j) \leq \pi f \sum_{j=1}^{n} r_j^2.$$

From Hölder's inequality, we have that

$$\left(\sum_{j=1}^{n} (r_j^2)^{3/2}\right)^{2/3} \left(\sum_{j=1}^{n} 1^3\right)^{1/3} \geq \sum_{j=1}^{n} r_j^2 \geq \frac{A}{\pi f},$$

$$\left(\frac{3}{4\pi} V\right)^{2/3} n^{1/3} \geq \frac{A}{\pi f},$$

$$n \geq \frac{A^3}{\pi^3 f^3}\left(\frac{4\pi}{3V}\right)^2 = \frac{16A^3/9}{\pi f^3 V^2}.$$

We see that we can take any $c = c(P)$ with $0 < c < 16A^3/9\pi f^3$.

Solution to B3 (based on a student paper). Yes. For a proof, first consider the special case in which for all i and j with $i > j$, team i defeated team j in their encounter. Then start by choosing team 2 from the round in which teams 1 and 2 played each other, and then go through the other teams $3, 4, \dots, n$ in order. Each of those teams, say i, has $i - 1$ victories, and when we get to team i, winners from only $i - 2$ rounds have been chosen, so it is possible to choose team i as the winner for a new round; all teams except team 1 will be chosen and no team will be chosen more than once, so we are done in this case.

It is now enough to show that if we can make a set of choices with the desired property in one tournament, then we can again do so if the outcome of a single game in the tournament is changed. We may assume that the game whose outcome is changed was between teams 1 and 2 and was originally won by team 1. If team 1 was not chosen in that round, we can keep the same choices. If team 2 was not chosen in any other round, we can choose team 2 in that round and keep all the other choices.

Otherwise, we will choose team 2 in that round, and in the round for which team 2 was originally chosen, we consider team 1's opponent, say team 3. If team 1 defeated team 3, or if team 3 was not chosen in any other round, we can choose the winner of the game between teams 1 and 3 and we will be done. Otherwise, we choose team 3 anyway, and consider team 1's opponent, say team 4, in the round for which team 3 was originally chosen. In general, if we have now chosen teams 2 through i in rounds in which they defeated team 1 and team i was originally chosen in a different round, then we label team 1's opponent in that different round as team $i+1$, and we choose the winner of the game between teams 1 and $i+1$. Eventually this process will terminate, either because team 1 defeated team $i+1$ or because team $i+1$ is the team that was originally not chosen in any round, and we are then done.

Solution to B4. Let $f(x) = x + e^{-x}$, so $a_{n+1} = f(a_n)$. Note that for $x > 0$, $f'(x) = 1 - e^{-x} > 0$, so f is increasing for $x \geq 0$. We now show by induction on n that $a_n > \log(n+1)$ for all n. The base case is clear, and $a_n > \log(n+1)$ implies

$$\begin{aligned} a_{n+1} = f(a_n) > f(\log(n+1)) &= \log(n+1) + \frac{1}{n+1} \\ &> \log(n+1) + \int_{n+1}^{n+2} \frac{1}{x}\, dx = \log(n+2), \end{aligned}$$

completing the induction.

It follows that

$$\begin{aligned} a_{n+1} - a_n = e^{-a_n} &< \frac{1}{n+1} \\ &< \int_n^{n+1} \frac{1}{x}\, dx = \log(n+1) - \log n, \end{aligned}$$

so $a_{n+1} - \log(n+1) < a_n - \log n$. Thus the $a_n - \log n$ form a decreasing sequence of positive numbers, so they have a limit. (It can be shown that the limit is 0.)

Solution to B5. Note that every function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = \sup_{t\in\mathbb{R}} \big(x h_1(t) + h_2(t)\big) \tag{2}$$

is convex, where $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are any two functions subject only to the condition that the supremum on the right-hand side exists for every $x \in \mathbb{R}$. Indeed, for every $x, y \in \mathbb{R}$ and every $\lambda \in (0, 1)$,

$$\begin{aligned} &\lambda f(x) + (1-\lambda) f(y) \\ &\quad = \sup_{t\in\mathbb{R}} \big(\lambda x h_1(t) + \lambda h_2(t)\big) + \sup_{t\in\mathbb{R}} \big((1-\lambda) y h_1(t) + (1-\lambda) h_2(t)\big) \\ &\quad \geq \sup_{t\in\mathbb{R}} \Big(\big((\lambda x + (1-\lambda)y\big) h_1(t) + \big(\lambda + (1-\lambda)\big) h_2(t)\Big) \\ &\quad = f\big(\lambda x + (1-\lambda) y\big), \end{aligned}$$

so that f is a convex function. We claim that the converse is also true, namely, every convex function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2) for some $h_1, h_2 : \mathbb{R} \to \mathbb{R}$; in fact, we claim that h_1, h_2 can be chosen so that the slightly stronger condition

$$f(x) = \max_{t\in\mathbb{R}} \bigl(xh_1(t) + h_2(t)\bigr) \tag{3}$$

holds. Indeed, since f is convex, we know that f is a continuous function with left and right derivatives at every point, which satisfies

$$f'_-(a) \le f'_+(a) \le \frac{f(b) - f(a)}{b - a} \le f'_-(b)$$

for every $a, b \in \mathbb{R}$ with $a < b$. From this it follows that

$$f(x) \ge (x - t)f'_-(t) + f(t)$$

for every $t \in \mathbb{R}$, with equality for $t = x$. It follows that (3) holds with

$$h_1(t) = f'_-(t), \quad h_2(t) = f(t) - tf'_-(t).$$

Let $g_1, g_2 : \mathbb{R} \to [1, \infty)$ be as in the statement of the problem. In the first part of the above discussion, we have proved that

$$f(x) = \sup_{t\in\mathbb{R}} \bigl(x \log g_1(t) + \log g_2(t)\bigr)$$

defines a convex function $f : \mathbb{R} \to \mathbb{R}$, since $\log g_1, \log g_2 : \mathbb{R} \to \mathbb{R}$ are bounded. But then the function $e^{f(x)}$ is also convex. This is well known and follows from

$$\lambda e^{f(x)} + (1-\lambda)e^{f(y)} \ge e^{\lambda f(x)+(1-\lambda)f(y)} \ge e^{f(\lambda x+(1-\lambda)y)},$$

where the first step uses the convexity of the exponential function, and the second step uses the convexity of f and the monotonicity of the exponential function. Since $e^{f(x)}$ is a convex function, it follows by (3) that

$$e^{f(x)} = \max_{t\in\mathbb{R}} \bigl(xh_1(t) + h_2(t)\bigr)$$

for some $h_1, h_2 : \mathbb{R} \to \mathbb{R}$. This is equivalent to what was to be proved.

Solution to B6. Consider $a \ne 0, 1, -1$, the three classes fixed by π. The cycle containing a has the same length as the cycle containing $-a \not\equiv a \pmod p$. Thus, the parity of π is determined by those cycles containing both a and $-a$. Similarly, the cycle containing a has the same length as the cycle containing $a^{-1} \not\equiv a \pmod p$. Thus we are down to cycles containing a, $-a$, and a^{-1}. Then if it takes k applications of π to get from a to $-a$, the cycle will have length $2k$; on the other hand, the same argument applies to a^{-1} instead of $-a$, so $-a \equiv a^{-1} \pmod p$, that is, $a^2 \equiv -1 \pmod p$. For such a, $a^3 \equiv -a \pmod p$ and so $k = 1$. Because the multiplicative group $\pmod p$ is cyclic of order $p - 1$, or by Euler's criterion, there are no such a when $p \equiv 3 \pmod 4$ and there are two that form a cycle of order 2 when $p \equiv 1 \pmod 4$. Therefore, π is even in the former case and odd in the latter.

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CONTENTS

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS